Math 4220: Homework 1

This is the first homework assignment for Math 4220. I have broken the homework assignment into two parts. Part I has exercises that you should do (and I expect you to do) but you need not turn in. The exercises in the second part of the homework are exercises you should write up and submit via Gradescope. Your solutions should be consistent with the directions in the syllabus. If you get stuck on any part of the homework, please come and see me or Max. More importantly, have fun!

Part I (Do not write up)

Exercise 1. Read Pages 26-56 of the course textbook.

Exercise 2. Do Exercises 1, 3, 5, 7, 11, 16, and 17 in Section 1.4 of the textbook.

Exercise 3. Do Exercises 5, 6, 10, and 18 in Section 1.5 of the textbook.

Exercise 4. Do Exercises 1, 2-8, 11, 15, 16, 17, 19, and 21 in Section 1.6 of the textbook.

Exercise 5. Do Exercises 1, 3, 4, 5, 8, 9, 11, and 13 in Section 2.1 of the textbook.

Part II (Write up and Submit via Gradescope)

Exercise 1 (The complex exponential). In this exercise, we shall play around with the complex exponential where we (mostly) consider only purely imaginary arguments[1] $z = i\theta$ or $z = i\omega t$.

1. We used the following fact several times in lecture:

   Fact 1. *For a complex number z, $|z| = 1$ if and only if $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.*

   Prove this fact. Also say whether or not θ is unique.

2. In view of the above fact, for a real number ω, we can view $f(t) = e^{i\omega t}$ as a function from $\mathbb{R}$ into the unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$. If you put your multivariable calculus "hat" on, you will recognize that $f(t)$ is a *parameterization* of the unit circle S whenever $\omega \neq 0$. Write down this parameterization in the language of multivariable calculus[2], describe the direction in which f parameterizes the circle (clockwise or counterclockwise) and determine the parameterization's velocity and speed. You will should find that the direction, velocity, and speed all depend on ω.

3. Sketch the curves

   (a) $f_1(t) = e^{1+it}$ for $t \in (-\pi, \pi]$.

   (b) $f_2(t) = e^{-1+it}$ for $t \in (-\pi, \pi]$.

   (c) $f_3(t) = e^{t+i}$ for $t \in \mathbb{R}$

   (d) $f_4(t) = e^{t+ti}$ for $t \in \mathbb{R}$.

Exercise 2. Please do Exercise 17 in Section 1.5 of the textbook.

Exercise 3 (A little multivariable calculus). Consider the function

$$u(x, y) = \begin{cases} 2 & |z| < 1 \\ -2 & \mathrm{Re}(z) > 2 \end{cases}$$

which is defined on the open set $D = \{z = x + iy : |z| < 1 \text{ or } \mathrm{Re}(z) = x > 2\}$.

1. Sketch D.

[1] Here, I'm using argument to mean the input variable of a function which is, in general, different from the meaning of $\arg(z)$. This reminds me of a famous quote by H. Poincare: "Mathematics is the art of giving the same names to different things" (See here); this quote is paralleled by a related one about poetry.

[2] In multivariable calculus, you might have written $f(t) = \mathbf{r}(t) = (x(t), y(t))$ for the parameterization where $x(t)$ and $y(t)$ are real-valued functions. In this case, the parameterization's velocity is $\dot{r}(t) = (\dot{x}(t), \dot{y}(t))$ and its speed is $\|\dot{r}(t)\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$ where the "dot" denotes the time derivative.

2. Show that $\partial u/\partial x$ and $\partial u/\partial y$ are both (identically) zero on D.

3. Explain why this does not contradict Theorem 1.

Exercise *4.* Please do Exercise 21 of Section 1.6. In the course of this exercise, make sure to give clear and correct reasoning for why your answer must represent all such functions $u(x, y)$.

Exercise *5.* Please do Exercise 8 in Section 2.1 of the textbook.

Exercise *6.* Please do Exercise 12 in Section 2.1 of the textbook.