MATH 4220 HOMEWORK 1 SOLUTIONS

Exercise 1
1. For any $\theta \in \mathbb{R}$, we have $|e^{i\theta}|^2 = |\cos(\theta) + i\sin(\theta)|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$. Conversely, suppose $z \in \mathbb{C}$, so that we can write $z = re^{i\theta}$ for some $r > 0$ and $\theta \in \mathbb{R}$. Then $1 = |z| = r$. Finally, θ is not unique: $e^{i\theta} = e^{i(\theta+2\pi k)}$ for any $k \in \mathbb{Z}$.

2. In multivariable calculus notation, $f(t) = (cos(\omega t), \sin(\omega t))$, which is a counter-clockwise parametrization if and only if $\omega > 0$. The velocity is

$$f'(t) = (-\omega \sin(\omega t), \omega \cos(\omega t)),$$

and the speed is

$$||f'(t)|| = \sqrt{\omega^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)} = |\omega|.$$

3. (a) This is a circle with radius e.
(b) This is a circle with radius e^{-1}.
(c) This is a slanted ray.
(d) This is a spiral.

Exercise 2
Set $\alpha := 1 + \omega_m^l + \cdots + \omega_m^{(m-1)l}$. Then

$$\begin{aligned}\omega_m^l \alpha &= \omega_m^l + \omega_m^{2l} + \cdots + \omega_m^{(m-1)l} + \omega_m^{ml} \\ &= \omega_m^l + \omega_m^{2l} + \cdots + \omega_m^{(m-1)l} + 1 \\ &= \alpha.\end{aligned}$$

In other words, $\alpha(1-\omega_m^l) = 0$, but because l is not divisible by m, we have $l/m \notin \mathbb{Z}$, hence

$$\omega_m^l = e^{2\pi i \frac{l}{m}} \neq 1.$$

This means that $\alpha = 0$.

Exercise 3
D consists of a disjoint circle of radius 1 and a half-plane containing all points (strictly) to the right of $x = 2$. The partial derivatives of u are zero because the functions are locally constant (so the difference quotients are all zero when $|h|$ is small. This does not contradict Theorem 1 because D is not a domain (in particular, it is not connected).

Exercise 4
Suppose u is a real-valued function satisfying

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

in the annulus A.

Method 1: Integrate in x to get (for x near 2)

$$u(x,0) = u(2,0) + \int_2^x \frac{\partial u}{\partial x}(x',0)dx' = \log((x')^2)\big|_{x'=2}^{x'=x} = u(2,0) + \log(x^2) - \log(4).$$

Setting $a := u(2,0) - \log(4)$, we then integrate in y (for y near 0) to get

$$\begin{aligned} u(x,y) - u(x,0) &= \int_0^y \frac{\partial u}{\partial y}(x,y')dy' = \log(x^2+(y')^2)\big|_{y'=0}^{y'=y} \\ &= \log(x^2+y^2) - \log(x^2). \end{aligned}$$

Combining equations gives $u(x,y) = \log(x^2+y^2) + a$. While we have only shown that this holds near $(2,0)$, we observe that this function is well-defined on all of A, and has the prescribed partial derivatives. If v is another function with the same partial derivatives, then the partial derivatives of $u - v$ are all zero, so $u - v$ is constant since A is connected.
(Note that the choice of $(2,0)$ as a basepoint for our initial integration was only for the sake of convenience, and in fact we could have chosen any $(x_0, y_0) \in A$.

Method 2: In polar coordinates, since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$\frac{\partial u}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial u}{\partial y} = \cos(\theta)\frac{2r\cos(\theta)}{r^2} + \sin(\theta)\frac{2r\sin(\theta)}{r^2} = \frac{2}{r},$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial u}{\partial y} = -r\sin(\theta)\frac{2r\cos(\theta)}{r^2} + r\cos(\theta)\frac{2r\sin(\theta)}{r^2} = 0.$$

Fixing r and integrating in θ, we see that $\theta \mapsto u(r,\theta)$ is constant. That is, u is radial, so is determined by its values along a single ray. We integrate from $r = 2$ to get

$$u(r,\theta) = u(1,\theta) + \int_2^r \frac{\partial u}{\partial r}(s,\theta)ds = \log(r^2) - \log(4).$$

The function is thus determined by a choice of $a = u(1,\theta) - \log(4)$ for some choice of angle θ, and $u(r,\theta) = a + \log(r^2)$.
Note that in this method, we implicitly used that A is connected, since our paths of integration connected any point of the annulus to the point $(2,0)$.

Method 3: Guess the function u, and note that any other function v with the same partial derivatives must differ by a constant since the partial derivatives of $u - v$ are all zero.

Exercise 5 (a) This is the upper semi-disk rotated counter-clockwise by 45 degrees (or $\pi/4$ radians.
(b) The same as (a), but clockwise.

(c) The same as (a), but 135 degree (or $3\pi/4$ radians).

Exercise 6 (Note there are some small inaccuracies in this exercise because it seems to assume $a \neq 0$. In addition, a function of the form $f(z) = az + b$ is usually called an *affine transformation*, with the terminology *linear transformation* reserved for the case where $b = 0$.)
Write $a = re^{i\theta}$. Define $F(z) := z + b$, $G(z) := e^{i\theta}z$, and $H(z) := rz$. Then F is a translation, G is a rotation, and H is a magnification (more commonly called a dilation). Moreover,

$$(F \circ G \circ H)(z) = (G \circ H)(z) + b = e^{i\theta}G(z) + b = re^{i\theta}z + b = az + b.$$

Clearly, translations preserve lines and circles. A rotation takes lines to lines with different angles, and circles to circles with possibly different centers. A magnification takes a circle to a circle with possibly different radius, and a line to a parallel line. These statements are all geometrically obvious (though they can be verified, for example, by fixing a parametrization, or an equation whose solution is each shape.)