Math 4220: Homework 2

This is the second homework assignment for Math 4220. I have broken the homework assignment into two parts. Part I has exercises that you should do (and I expect you to do) but you need not turn in. The exercises in the second part of the homework are exercises you should write up and submit via Gradescope. Your solutions should be consistent with the directions in the syllabus. If you get stuck on any part of the homework, please come and see me or Max. More importantly, have fun!

Part I (Do not write up)

Exercise 1. Read Sections 2.2 and 2.3 of the course textbook.

Exercise 2. Do Exercises 1, 2, 3, 5, 7, 8, 9, 11, 13, 21, and 25 in Section 2.2 of the textbook.

Exercise 3. Do Exercises 1, 2, 3, 7, 9, 11, 12, and 13 in Section 2.3 of the textbook.

Part II (Write up and Submit via Gradescope)

Exercise 1 (Using the definition of the limit). In this exercise, you will use the following so-called ϵ-N definition of the limit (which we saw in class):

Definition 1. *Let $\{z_n\}$ be a sequence of complex numbers and let z_0 be a complex number. We say that the sequence $\{z_n\}$ converges to z_0 and write $\lim_{n\to\infty} z_n = z_0$ (or equivalently) $z_n \to z_0$ as $n \to \infty$) provided that, for all $\epsilon > 0$ there is a natural number N for which*

$$|z_n - z_0| < \epsilon \qquad \text{whenever} n \geq N.$$

Intuitively, we can understand this definition in the following way: The sequence $\{z_n\}$ converges to z_0 if, for every ϵ-neighborhood of z_0, $D_\epsilon(z_0)$, the elements of the sequence are eventually (all) in $D_\epsilon(z_0)$. Please use the above ϵ-N definition to prove:

1. If $|z| < 1$, then the sequence $\{z_n = z^n\}$ converges to $z_0 = 0$. Note: This one is very similar to the example I did in class.
2. $\lim_{n\to\infty} e^{i/n} = 1$. Hint: Feel free to use the inequalities $|\sin(x)| < |x|$ and $|\cos(x) - 1| < x^2$.
3. If $\lim_{n\to\infty} z_n = 1 + i$, then $\lim_{n\to\infty}(6z_n - 4) = 2 + 6i$.

Exercise 2 (Connecting Limits to Multivariable Calculus). In multivariable calculus, we learned how to take limits of real-valued functions of two (and more) variables. With the limit, we were able to talk about derivatives and continuity. Let us recal this definition (which you should observe is very similar to that given for complex-valued functions of a complex variable).

Definition 2. *Let $u : \mathcal{D} \to \mathbb{R}$ where $D \subseteq \mathbb{R}^2$. Suppose that there is a positive real number $\rho > 0$ for which $\{(x, y) \in \mathbb{R}^2 : 0 < |(x, y) - (x_0, y_0)| < \rho\} \subseteq \mathcal{D}$ (this means that u is defined on a "punctured" neighborhood of (x_0, y_0).). Given a real number u_0, we say that the limit of $u(x, y)$ is u_0 as (x, y) approaches (x_0, y_0) and write*

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x, y) = u_0$$

if, for every $\epsilon > 0$ there exists $\delta > 0$ for which

$$|u(x, y) - u_0| < \epsilon \qquad \text{whenever} \qquad 0 < |(x, y) - (x_0, y_0)| < \delta.$$

In the context of this definition $|\cdot|$ stands for the Euclidean norm and so $|(x, y) - (x_0, y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

1. Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Prove that

$$\lim_{z\to z_0} f(z) = w_0$$

if and only if

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x, y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x, y) = v_0.$$

2. Use the statement you just proved to show that $f(z) = e^z$ is continuous everywhere.

Exercise 3. In class, I stated the following characterization of limits of functions in terms of sequences.

Proposition 3. *Let* z_0 *be a complex number and* $f : \mathcal{D} \to \mathbb{C}$ *be defined on a (punctured) neighborhood of* z_0*. Then* $\lim_{z\to z_0} f(z) = w_0$ *if and only if, for every sequence* $\{z_n\} \subseteq \mathcal{D}$ *for which* $\lim_{n\to\infty} z_n = z_0$*,* $\lim_{n\to\infty} f(z_n) = w_0$*.*

In this exercise, you will prove half of the proposition above. In particular, please prove the statement:

> If for every sequence $\{z_n\} \subseteq \mathcal{D}$ for which $\lim_{n\to\infty} z_n = z_0$ we have $\lim_{n\to\infty} f(z_n) = w_0$, then $\lim_{z\to z_0} f(z) = w_0$.

Hint: It is easy to prove the contrapositive statement, i.e., if you assume that the $\lim_{z\to z_0} f(z) \neq w_0$, you can then show that the sequence property does not hold.

Exercise 4 (Infinite Limits). Consider the following two definitions (which can be found on Page 62 of your textbook):

Definition 4. *Let* $\{z_n\}$ *be a sequence of complex numbers. We say that* $\{z_n\}$ *diverges to* ∞ *and write* $z_n \to \infty$ *if, for all (think large)* $M > 0$*, there is a natural number* N *for which* $|z_n| \geq M$ *whenever* $n \geq N$*.*

Definition 5. *Let* $f(z)$ *be defined on a punctured neighborhood of* z_0*. We say that* $f(z)$ *diverges to* ∞ *and write* $\lim_{z\to z_0} f(z) = \infty$ *if, for every* $M > 0$ *there is a* $\delta > 0$ *for which* $|f(z)| \geq M$ *whenever* $n \geq N$*.*

1. Come up with your own definition of the notion "$\lim_{z\to\infty} f(z) = \infty$." Your definition should be precise like those above.
2. Use your definition to show that
$$\lim_{z\to\infty} z^2 = \infty.$$
3. Argue, using your definition, that $\lim_{z\to\infty} e^z \neq \infty$. Weird, right?
4. Using your definition, show that:
$$\text{If} \qquad \lim_{z\to\infty} f(z) = \infty, \qquad \text{then} \qquad \lim_{w\to 0} \frac{1}{f(1/w)} = 0.$$
5. Is the converse the to the above statement also true?

Exercise 5. Please do Exercise 4 in Section 2.3 of the course textbook.

Exercise 6. Please do Exercise 8 in Section 2.3 of the course textbook.