MATH 4220 HOMEWORK 2 SOLUTIONS

Exercise 1

(1) Without loss of generality, we can assume $z \neq 0$. Let $\epsilon > 0$, and set $a := |z|$. Then, for any $n \in \mathbb{N}$, we have $|z_n - 0| = a^n$, so for any $n \geq \log_a(\epsilon)$, we have

$$a^n = a^{n-\log_a(\epsilon)}\epsilon < \epsilon$$

since $0 \leq a < 1$. We may therefore take N to be any positive integer which is at least $\log_a(\epsilon)$.

(2) Let $\epsilon > 0$. For any $n \in \mathbb{N}$,

$$|e^{\frac{i}{n}} - 1|^2 = |1 - \cos(1/n)|^2 + |\sin(1/n)|^2 \leq \frac{1}{n^4} + \frac{1}{n^2} \leq \frac{2}{n^2}.$$

Thus, if take N to be any integer greater than $\frac{\sqrt{2}}{\epsilon}$, then whenever $n \geq N$, we have $|e^{\frac{i}{n}} - 1| < \epsilon$.

(3) Fix $\epsilon > 0$. Because $\lim_{n\to\infty} z_n = 1 + i$, we can find $N \in \mathbb{N}$ such that $|z_n - (1 + i)| < \frac{1}{6}\epsilon$ whenever $n \geq N$. Then, whenever $n \geq N$, we have

$$|(6z_n - 4) - (2 + 6i)| = 6|z_n - (1 + i)| < 6 \cdot \frac{\epsilon}{6} = \epsilon.$$

Exercise 2:

(1) We write $z = x+iy$ and $z_0 = x_0+iy_0$. First suppose $\lim_{z\to z_0} f(z) = w_0$, and fix $\epsilon > 0$. Then we can find $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$. It follows that

$$|u(x, y) - u_0| \leq \sqrt{|u(x, y) - u_0|^2 + |v(x, y) - v_0|^2} = |f(z) - w_0| < \epsilon,$$

whenever $0 < |z-z_0| < \delta$, but $|(x, y)-(x_0, y_0)| = |z-z_0|$, so $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$. The proof that $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$ is similar.

Conversely, suppose $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. Fix $\epsilon > 0$, and then choose $\delta > 0$ such that $|u(x_0, y_0) - u_0| < \frac{\epsilon}{2}$ and $|v(x_0, y_0) - v_0| < \frac{\epsilon}{2}$ whenever $0 < |(x, y) - (x_0, y_0)| < \delta$. Then

$$|f(z) - w_0| = \sqrt{|u(x, y) - u_0|^2 + |v(x, y) - v_0|^2} < \sqrt{\frac{\epsilon^2}{4} + \frac{\epsilon^2}{4}} = \frac{\epsilon}{\sqrt{2}} < \epsilon$$

whenever $0 < |(x, y) - (x_0, y_0)| < \delta$ (again using that $|(x, y) - (x_0, y_0)| = |z - z_0|$), so $\lim_{z\to z_0} f(z) = w_0$.

(2) By part (1), it suffices to show note that the real part $u(x, y) = e^x \cos(y)$ and imaginary part $v(x, y) = e^x \sin(y)$ are continuous as functions $\mathbb{R}^2 \to \mathbb{R}$. In fact, this implies

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = e^{x_0} \cos(y_0),$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = e^{x_0}\sin(y_0),$$

so part (1) implies

$$\lim_{z\to z_0} f(z) = e^{x_0}\cos(y_0) + ie^{x_0}\sin(y_0) = f(z_0).$$

Exercise 3: Suppose $\lim_{z\to z_0} f(z) \neq w_0$. This means there exists $\epsilon > 0$ such that, for any $\delta > 0$, there is a point z with $0 < |z - z_0| < \delta$ such that $|f(z) - w_0| \geq \epsilon$. In particular, we can choose $\delta = 1/j$ for any $j \in \mathbb{N}$, obtaining points z_j with $0 < |z_0 - z_j| < 1/j$ but $|f(z_j) - w_0| \geq \epsilon$. In particular, the sequence (z_j) satisfies $\lim_{j\to\infty} z_j = z_0$, but $\lim_{j\to\infty} f(z_j) \neq w_0$.

Note: One way of finding the negation of the statement $\lim_{z\to z_0} f(z_0) = w$ is to first write this statement in terms of logical quantifiers:

$$\forall\epsilon > 0 : \exists\delta > 0 : \forall z \text{ with } 0 < |z - z_0| < \delta : |f(z) - f(z_0)| < \epsilon.$$

Recall that $\forall$ means for every, $\exists$ means there exists, and $\neg$ means the negation of a logical statement. These logical operators satisfy the relations $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$. Thus, if P is the logical statement $\lim_{z\to z_0} f(z_0) = w$, then the negation $\neg P$ is given by the following logically equivalent statements:

$$\neg\left(\forall\epsilon > 0 : \exists\delta > 0 : \forall z \text{ with } 0 < |z - z_0| < \delta : |f(z) - f(z_0)| < \epsilon\right),$$

$$\exists\epsilon > 0 : \neg\left(\exists\delta > 0 : \forall z \text{ with } 0 < |z - z_0| < \delta : |f(z) - f(z_0)| < \epsilon\right),$$

$$\exists\epsilon > 0 : \forall\delta > 0 : \neg\left(\forall z \text{ with } 0 < |z - z_0| < \delta : |f(z) - f(z_0)| < \epsilon\right),$$

$$\exists\epsilon > 0 : \forall\delta > 0 : \exists z \text{ with } 0 < |z - z_0| < \delta : |f(z) - f(z_0)| \geq \epsilon.$$

You do not need to use all this formalism to get the right answer, but this method provides an algorithmic way of finding the negation of any logical statement, no matter how complicated.

Exercise 4: Note: there is a typo in the statement of this problem. In definition 5, "whenever $n \geq N$" should be replaced by "whenever $|z - z_0| < \delta$". However, this definition is written correctly (on page 62) in the textbook, which you read as part of exercise 1.

(1) For any $M > 0$, there exists $r > 0$ such for any $z \in \mathbb{C}$ satisfying $|z| \geq r$, we have $|f(z)| \geq M$.

(2) Fix $M > 0$, and take $r := \sqrt{M}$. Then, for any z satisfying $|z| \geq r$, we have

$$|z^2| = |z|^2 \geq r^2 \geq M.$$

(3) Let $M := 2$, and let $r > 0$ be arbitrary. Then we $z = ir$ satisfies $|z| \geq r$, yet $|f(z)| = |e^{ir}| = 1 < 2$.

(4) Fix $\epsilon > 0$. Then we can find $r > 0$ such that $|f(z)| \geq 2\epsilon^{-1}$ for all z with $|z| \geq r$. Thus, for any $w \in \mathbb{C}$ with $0 < |w| < r^{-1}$, we have $|1/w| \geq r$, so $|f(1/w)| \geq 2\epsilon^{-1}$, hence

$$\left|\frac{1}{f(1/w)}\right| \leq \frac{\epsilon}{2}.$$

(5) Yes. Suppose $\lim_{w\to 0} \frac{1}{f(1/w)} = 0$, and fix $M > 0$. Then we can find $\delta > 0$ such that for any w with $0 < |w| < \delta$, we have

$$\left|\frac{1}{f(1/w)}\right| \leq M^{-1},$$

or equivalently $|f(1/w)| \geq M$. If we set $r := \delta^{-1}$, then for any z with $|z| \geq r$, we have $0 < |1/z| < \delta$, hence $|f(z)| = |f(1/(1/z))| \geq M$.

Exercise 5:
(a) Suppose by way of contradiction that there exists $z_0 \in \mathbb{C}$ such that $L := (\mathrm{Re})'(z_0))$ exists. Then there exists $\delta > 0$ such that for any $h \in \mathbb{C}$ with $|h| < \delta$, we have

$$\left|\frac{\mathrm{Re}(z_0 + h) - \mathrm{Re}(z_0)}{h} - L\right| < \frac{1}{4}.$$

In particular, for any $t \in \mathbb{R}$ satisfying $|t| < \delta$, we have

$$\frac{1}{4} > \left|\frac{\mathrm{Re}(z_0 + t) - \mathrm{Re}(z_0)}{t} - L\right| = |1 - L|,$$

so that

$$|L| \geq 1 - |L - 1| \geq \frac{3}{4}.$$

On the other hand,

$$\frac{1}{4} > \left|\frac{\mathrm{Re}(z_0 + it) - \mathrm{Re}(z_0)}{it} - L\right| = |L|,$$

so that $\frac{3}{4} < |L| < \frac{1}{4}$, which is impossible.

(b) We argue by contradiction as above, choosing $\delta > 0$ such that for any $h \in \mathbb{C}$ with $|h| < \delta$, we have

$$\left|\frac{\mathrm{Im}(z_0 + h) - \mathrm{Im}(z_0)}{h} - L\right| < \frac{1}{4}.$$

For any $t \in \mathbb{R}$ with $|t| < \delta$, we have

$$\frac{1}{4} > \left|\frac{\mathrm{Im}(z_0 + t) - \mathrm{Im}(z_0)}{t} - L\right| = |L|,$$

and

$$\frac{1}{4} > \left|\frac{\mathrm{Im}(z_0 + t) - \mathrm{Im}(z_0)}{t} - f'(z_0)\right| = |1 - L|,$$

so we establish a contradiction as in part (a).

(c) Suppose by way of contradiction that $L := |\cdot|'(z_0)$ exists for some $z_0 \in \mathbb{C}$. If $z_0 = 0$, then (letting t denote a real number)

$$-1 = \lim_{t\to 0^-} \frac{|t| - |0|}{t} = L = \lim_{t\to 0^+} \frac{|t| - |0|}{t} = 1,$$

which is impossible. We may thus assume $z_0 \neq 0$. Then

$$L = \lim_{t\to 0} \frac{|z_0 + tz_0| - |z_0|}{tz_0} = \frac{|z_0|}{z_0} \lim_{t\to 0} \frac{1+t}{t} = \frac{|z_0|}{z_0} \neq 0.$$

On the other hand,

$$L = \lim_{t\to 0} \frac{|z_0 + itz_0| - |z_0|}{itz_0} = \frac{|z_0|}{iz_0} \lim_{t\to 0} \frac{\sqrt{1+t^2} - 1}{t} = 0,$$

a contradiction.

Exercise 6 Fix $\epsilon > 0$. Because $f'(z_0) \neq 0$ exists, we can find $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \min(\epsilon, \frac{1}{2}|f'(z_0)|)$$

whenever $0 < |z - z_0| < \delta$. Then the triangle inequality implies

$$\left| \frac{|f(z) - f(z_0)|}{|z - z_0|} - |f'(z_0)| \right| = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right|,$$

so because $\epsilon > 0$ was arbitrary, the first claim is true. Next, note that whenever $0 < |z - z_0| < \delta$, we have

$$\begin{aligned}
\mathrm{Re}\left(f'(z_0) \frac{f(z) - f(z_0)}{z - z_0} \right) &= |f'(z_0)|^2 + \mathrm{Re}\left(f'(z_0) \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right) \right) \\
&\geq |f(z_0)|^2 - |f'(z_0)| \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \\
&\geq \frac{1}{2}|f'(z_0)| > 0.
\end{aligned}$$

In particular, the difference quotients when $0 < |z - z_0| < \delta$ all lie in a single half-plane, so we can choose a branch of arg which is defined for all such z. Next, we observe that

$$\begin{aligned}
\mathrm{cis}\left(\arg\left(f(z) - f(z_0)\right) - \arg(z - z_0)\right) &= \mathrm{cis}\left(\arg\left(\frac{f(z) - f(z_0)}{z - z_0} \right) \right) \\
&= \left| \frac{f(z) - f(z_0)}{z - z_0} \right|^{-1} \frac{f(z) - f(z_0)}{z - z_0}.
\end{aligned}$$

Taking the limit as $z \to z_0$, and using the first part of this exercise gives

$$\lim_{z\to z_0} \mathrm{cis}\left(\arg\left(f(z) - f(z_0)\right) - \arg(z - z_0)\right) = \mathrm{cis}\left(\arg(f'(z_0))\right).$$

Observe that the inverse of the exponential map is continuous on any branch where it is defined, so the second claim follows.