MATH 4220 HOMEWORK 3 SOLUTIONS

Exercise 1: Because $f(x) = f(iy) = 0$ for all $x, y \in \mathbb{R}$, we have

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0,$$

so the Cauchy-Riemann equations hold at $z = 0$. However, the difference quotient along the line $x = y$ limits to

$$\lim_{t \to 0} \frac{f(t+it) - f(0,0)}{t+it} = \lim_{t \to 0} \frac{1}{t+it} \frac{t^3 + it^3}{2t^2} = \frac{1}{2},$$

but we noted that the directional derivative along the real axis is zero, so f is not differentiable at $z = 0$.

Exercise 2:
Method 1: (Chain Rule) Using $x = r\cos(\theta)$, $y = r\sin(\theta)$, we compute

$$\frac{\partial u}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial u}{\partial y} = \cos(\theta)\frac{\partial u}{\partial x} + \sin(\theta)\frac{\partial u}{\partial y},$$

$$\begin{aligned}\frac{\partial v}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial v}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial v}{\partial y} = -r\sin(\theta)\frac{\partial v}{\partial x} + r\cos(\theta)\frac{\partial v}{\partial y} \\ &= r\sin(\theta)\frac{\partial u}{\partial y} + r\cos(\theta)\frac{\partial u}{\partial x},\end{aligned}$$

where the last equality follows from the usual Cauchy-Riemann equations. Thus $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$. The other equation is proved similarly.

Method 2: (Difference quotients) Assume f is C^1 and satisfies the Cauchy-Riemann equations. Then f is analytic, so we should have (writing $z_0 = r_0 e^{i\theta_0}$)

$$\begin{aligned}f'(z_0) &= \lim_{r \to r_0} \frac{f(re^{i\theta_0}) - f(r_0 e^{i\theta_0})}{(r - r_0)e^{i\theta_0}} = e^{-i\theta_0} \lim_{r \to r_0} \frac{f(r, \theta_0) - f(r_0, \theta_0)}{r - r_0} \\ &= e^{-i\theta_0}\left(\frac{\partial u}{\partial r}(z_0) + i\frac{\partial v}{\partial r}(z_0)\right),\end{aligned}$$

but also

$$\begin{aligned}f'(z_0) &= \lim_{\theta \to \theta_0} \frac{f(r_0 e^{i\theta}) - f(r_0 e^{i\theta_0})}{r_0(e^{i\theta} - e^{i\theta_0})} = \frac{1}{r_0} \lim_{\theta \to \theta_0} \frac{f(r_0, \theta) - f(r_0, \theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} \\ &= \frac{1}{r_0}\left(\frac{\partial u}{\partial \theta}(z_0) + i\frac{\partial v}{\partial \theta}(z_0)\right)\frac{1}{ie^{i\theta_0}}\end{aligned}$$

Note that the last equality follows from the formula for the derivative of the complex exponential function. We write $f(r, \theta)$ for f viewed as a function of the two real variables r, θ. By equating these two expressions for $f'(z_0)$, we arrive at the desired formulas.

Exercise 3: Note that any line can be mapped onto the real axis by a transformation of the form $\alpha(z) := az + b$, where $a, b \in \mathbb{C}$, and $a \neq 0$. By the chain rule, $g := \alpha \circ f$ is also analytic, with $g'(z) = af'(z)$. Writing $g = u + iv$, we have $v(z) = 0$ for all $z \in D$, so the Cauchy-Riemann equations give $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ on D. Since D is connected, we conclude that g is constant, hence so is $f = a^{-1}g$.

Exercise 4:
(1) Writing $f = u + iv$, a direct computation (using $\Delta u = \Delta v = 0$) gives

$$
\begin{aligned}
2\Delta \log|f(z)| =& \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log(u^2+v^2) \\
=& \frac{\partial}{\partial x}\left(\frac{1}{u^2+v^2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{u^2+v^2}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)\right) \\
=& -\frac{2}{(u^2+v^2)^2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 - \frac{2}{(u^2+v^2)^2}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2 \\
&+ \frac{1}{u^2+v^2}\left(u\Delta u + v\Delta v + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) \\
=& -\frac{2}{(u^2+v^2)^2}\left(u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right. \\
&\qquad\left. + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2\right) \\
&+ \frac{1}{u^2+v^2}\left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right).
\end{aligned}
$$

We now apply the Cauchy-Riemann equations and cancel terms:

$$
\begin{aligned}
2\Delta \log|f(z)| =& -\frac{1}{(u^2+v^2)^2}\left(u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial y}\right)^2 + u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2\right) \\
&+ \frac{2}{u^2+v^2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right). \\
=& 0.
\end{aligned}
$$

(2) Define $u(x,y) := \frac{1}{\log(2)}\log|z|$, which is smooth and harmonic except at $z = 0$. This clearly satisfies $u = 0$ on S_1 and $u = 1$ on S_2.

Exercise 5: (1) Linear functions are harmonic, so $u(x,y) := 2 + 8x$ solves the boundary value problem.

(2) The Cauchy-Riemann equations imply v must satisfy

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 8, \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

The latter condition and the fundamental theorem of calculus give $v(x, y) = v_0(x)$ for some function v_0, and the former condition tells us $v_0'(y) = 8$, hence $v_0(y) = 8y + a$ for some $a \in \mathbb{R}$. Then $f(z) = 2 + ia + 8x + 8iy = (2 + ia) + 8z$ is analytic for any $a \in \mathbb{C}$, with $\text{Re}(f) = u$.

Exercise 6: This does not violate the maximum principle because the function is only harmonic away from the heat source.