Math 4220: Homework 4

This is the forth homework assignment for Math 4220. I have broken the homework assignment into two parts. Part I has exercises that you should do (and I expect you to do) but you need not turn in. The exercises in the second part of the homework are exercises you should write up and submit via Gradescope. Your solutions should be consistent with the directions in the syllabus. If you get stuck on any part of the homework, please come and see me or Max. More importantly, have fun!

Part I (Do not write up)

Exercise 1. Read Sections 3.1, 3.2, and 3.3 of the course textbook.

Exercise 2. Do Exercises 1, 2, 3, 5, 7, 9, 11, and 15 in Section 3.1 of the textbook.

Exercise 3. Do Exercises 1, 2, 3, 7, 9, 11, 12, 17, and 21 in Section 3.2 of the textbook.

Exercise 4. Do Exercises 1, 4, 5, and 9 in Section 3.3 of the textbook.

Part II (Write up and Submit via Gradescope)

Exercise 1. Please do Exercise 4 in Section 3.1 of the course textbook.

Exercise 2. Please do Exercise 16 in Section 3.1 of the course textbook.

Exercise 3. Please do Exercise 18 in Section 3.2 of the course textbook.

Exercise 4. Please do Exercise 22 in Section 3.2 of the course textbook.

Exercise 5. Please do Exercise 14 in Section 3.3 of the course textbook.

Exercise 6 (Power Functions). In this exercise, we shall explore how one should define z^α for a given complex number α.

1. Let's first focus on $\alpha = 1/n$ for some integer $n > 1$ and $z = x$ where x is a positive real number. Recall that ω is an nth root of x if $\omega^n = x$. Find all nth roots of x and, as a consequence, discuss how there is more than one way to define/make sense of $\sqrt[n]{x} = x^{\frac{1}{n}}$.

2. If
$$\operatorname{Log}(z) = \log(|z|) + i\operatorname{Arg}(z)$$
is the principal branch of logarithm with its domain $\mathcal{D} := \mathbb{C} \setminus \{0\}$ and "domain of analyticity"
$$\mathcal{D}^* := \{z \in \mathbb{C} : z \neq 0 \text{ and } -\pi < \operatorname{Arg}(z) < \pi\},$$
is it true that
$$e^{n\operatorname{Log}(\omega)} = x$$
for every ω you found in the previous Item? Please explain/justify your answer.

3. In studying your previous answer, you should note that there is one ω found in the first item for which
$$\operatorname{Log}(\omega) = \frac{1}{n}\operatorname{Log}(x) = \frac{1}{n}\log(x)$$
and, as necessary,
$$\left(e^{\frac{1}{n}\operatorname{Log}(x)}\right)^n = e^{\frac{n}{n}\operatorname{Log}(x)} = e^{\operatorname{Log}(x)} = x.$$
For this reason, we may define
$$x^{1/n} = e^{\frac{1}{n}\operatorname{Log}(x)}.$$
In slightly more generality, if z lives in the domain of the principal branch of the logarithm, i.e., $z \in \mathcal{D}$, we define the so-called principal branch of the nth root of z by
$$z^{1/n} = e^{\frac{1}{n}\operatorname{Log}(z)}.$$
Show that
$$\left(z^{\frac{1}{n}}\right)^n = z \tag{1}$$
for every $z \in \mathcal{D}$.

4. Show that $z \mapsto z^{1/2}$, though defined at $z = -1$, is not continuous at $z = -1$. Also, say where $z \mapsto z^{1/2}$ is continuous. Say where it is analytic.

5. Given your answer in the previous item, suppose that you wanted to define $z \mapsto z^{1/2}$ so that it was continuous at $z = -1$. To this end, using what you know about branches of the logarithm, find a reasonable way to define $z^{1/2}$ so that the resulting function is continuous in the left half of the plane, i.e., $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. For every z in this set, confirm that your definition still satisfies (1) (for $n = 2$).

6. Motivated by our work above, for any $\alpha \in \mathbb{C}$ and $z \in \mathcal{D} = \mathbb{C} \setminus \{0\}$, we define

$$z^\alpha = e^{\alpha \operatorname{Log}(z)}.$$

With this definition (and what you know of the exponential and principal branch of logarithm), do the following:

(a) Prove that, for any $\alpha, \beta \in \mathbb{C}$, and $z \in \mathcal{D}$,
$$z^\alpha z^\beta = z^{\alpha+\beta}.$$

(b) Prove that, for any $\alpha, \beta \in \mathbb{C}$, and $z \in \mathcal{D}$,
$$(z^\alpha)^\beta = z^{\beta\alpha}.$$

(c) Prove that, for any $\alpha \in \mathbb{C}$ and $z \in \mathcal{D}$,
$$z^{-\alpha} = \frac{1}{z^\alpha}$$

(d) For any $\alpha \in \mathbb{C}$, show that $z \mapsto z^\alpha$ is analytic on $\mathcal{D}^*$ compute
$$\frac{d}{dz} z^\beta.$$
for every $z \in \mathcal{D}^*$. Hint: You should simply your derivative formula as much as possible to see that it is exactly what it "should" be.