MATH 4220 HOMEWORK 4 SOLUTIONS

Exercise 1: By the fundamental theorem of algebra, we can write

$$p(z) = \lambda(z - b_1) \cdots (z - b_n).$$

By expanding and identifying coefficents of z^0, we arrive at $\lambda = 1$, and thus $a_0 = (-1)^n b_1 \cdots b_n$. If $|b_i| \leq 1$ for all i, then

$$1 < |a_0| = |b_1 \cdots b_n| \leq 1,$$

a contradiction.

Exercise 2: Write $R_{m,n}(z) = \frac{f(z)}{g(z)}$, where

$$f(z) = a_m z^m + \cdots + a_0, \qquad g(z) = b_n z^n + \cdots + b_0.$$

Then the triangle inequality gives

$$|f(z)| \leq |a_m| \cdot |z|^m + |a_{m-1}| \cdot |z|^{m-1} \cdots + |a_0| = |z|^m \left(|a_m| + |a_{m-1}| \cdot |z|^{-1} + \cdots |a_0| \cdot |z|^{-m} \right)$$

$$|g(z)| \geq |b_n| \cdot |z|^n - |b_{n-1}| \cdot |z|^{n-1} - \cdots - |b_0| = |z|^n \left(|b_n| - |b_{n-1}| \cdot |z|^{-1} - \cdots - |b_0| \cdot |z|^{-n} \right)$$

for all $z \in \mathbb{C} \setminus \{0\}$. If we choose

$$\Lambda \geq \frac{1}{2(m+n)} \max\{|a_0|, ..., |a_{m-1}|, |b_0|, ..., |b_{n-1}|\} \cdot \max\{|a_m|^{-1}, |b_n|^{-1}\},$$

then $g(z) \neq 0$, and

$$\frac{|f(z)|}{|g(z)|} \leq |z|^{m-n} \frac{2|a_m|}{\frac{1}{2}|b_n|}$$

whenever $|z| \geq \Lambda$. Similarly, when $|z| \geq \Lambda$, we have

$$\frac{|f(z)|}{|g(z)|} \geq |z|^{m-n} \frac{\frac{1}{2}|a_m|}{2|b_n|}.$$

Note: You did not need to be so explicit in choosing Λ – it would suffice to note that the terms in the parentheses approach $|a_m|$ and $|b_n|$, respectively (and that would also let you show that the constants c_1, c_2 can be taken arbitrarily close to $\frac{|a_m|}{|b_n|}$, though that was not asked for in the problem).

An alternative solution (with a less explicit choice of constants): By the fundamental theorem of algebra, let us write

$$R_{m,n}(z) = \frac{a_m(z - z_1)(z - z_2) \cdots (z - z_m)}{b_n(z - w_1)(z - w_2) \cdots (z - w_n)}$$

where a_m and b_n are non-zero. To estimate the modulus of this quantity, let's begin to estimate the quantity $|z - w|$ for a fixed $w \in \mathbb{R}$. Observe that, if $|z| \geq 2|w|$, we have $|w| \leq |z|/2$ and $-|w| \geq -|z|/2$. Using the triangle inequality, we find that

$$|z - w| \leq |z| + |-w| = |z| + |w| \leq |z| + |z|/2 = \frac{3}{2}|z|$$

whenever $|z| \geq 2|w|$. By the reverse triangle inequality, we have

$$|z-w| \geq ||z|-|w|| \geq |z|-|w| \geq |z|-|z|/2 = \frac{1}{2}|z|$$

whenever $|z| \geq 2|w|$. Putting these together, we see that

$$\frac{1}{2}|z| \leq |z-w| \leq \frac{3}{2}|z| \tag{0.1}$$

and

$$\frac{2}{3|z|} \leq \frac{1}{|z-w|} \leq \frac{2}{|z|} \tag{0.2}$$

whenever $|z| \geq 2|w|$ and $|z| > 0$. In view of these estimates, set

$$M = 1 + \max\{2|z_1|, 2|z_2|, \ldots, 2|z_m|, 2|w_1|, 2|w_2|, \ldots, 2|w_n|\}$$

(I added the 1 because I want to make sure $|z| > 0$ when $|z| > M$). Observe that

$$\begin{aligned} |R_{n,m}(z)| &= \left|\frac{a_m(z-z_1)(z-z_2)\cdots(z-z_m)}{b_n(z-w_1)(z-w_2)\cdots(z-w_n)}\right| \\ &= \left|\frac{a_m}{b_n}\right| |z-z_1||z-z_2|\cdots|z-z_m| \frac{1}{|z-w_1|}\frac{1}{|z-w_2|}\cdots\frac{1}{|z-w_n|} \end{aligned}$$

Upon putting together the estimates (0.1) and (0.2) applied to each term $|z-z_k|$ and $1/|z-w_k|$, we obtain

$$\left|\frac{a_m}{b_n}\right| 2^{n-m}3^{-n}|z|^{m-n} = \left|\frac{a_m}{b_n}\right| \left(\frac{|z|}{2}\right)^m \left(\frac{2}{3|z|}\right)^n \leq |R_{n,m}(z)|$$

and

$$|R_{n,m}(z)| \leq \left|\frac{a_m}{b_n}\right| \left(\frac{3|z|}{2}\right)^m \left(\frac{2}{|z|}\right)^n = \left|\frac{a_m}{b_n}\right| 2^{n-m}3^m|z|^{m-n}$$

whenever $|z| \geq M$. In other words,

$$c_1|z|^{m-n} \leq |R_{n,m}(z)| \leq c_2|z|^{m-n}$$

whenever $|z| \geq M$ where

$$0 < c_1 := \left|\frac{a_m}{b_n}\right| 2^{n-m}3^{-n} < \left|\frac{a_m}{b_n}\right| 2^{n-m}3^m := c_2.$$

Exercise 3: Using the hint,

$$\lim_{z\to 0}\frac{\sin(z)}{z} = \lim_{z\to 0}\frac{\sin(z)-\sin(0)}{z} = \frac{d}{dz}\bigg|_{z=0}\sin(z) = \cos(0) = 1,$$

$$\lim_{z\to 0}\frac{\cos(z)-1}{z} = \lim_{z\to 0}\frac{\cos(z)-\cos(0)}{z} = \frac{d}{dz}\bigg|_{z=0}\cos(z) = -\sin(0) = 0.$$

Exercise 4: Using the hint, we proceed by induction on m, with the case $m = 1$ immediate since $e^{\lambda_1 z}$ is not the zero function.. Suppose the claim holds for some $m \geq 1$, and assume $\lambda_1, ..., \lambda_{m+1} \in \mathbb{C}$ are distinct complex numbers such that

$$0 = c_1e^{\lambda_1 z} + \cdots + c_{m+1}e^{\lambda_{m+1}z}$$

for all $z \in \mathbb{C}$. Dividing both sides by $e^{\lambda_{m+1}z}$ and rearranging gives

$$-c_{m+1} = c_1e^{(\lambda_1-\lambda_{m+1})z} + \cdots + c_me^{(\lambda_m-\lambda_{m+1})z}$$

for all $z \in \mathbb{C}$. Next, we differentiate to obtain

$$0 = (\lambda_1 - \lambda_{m+1})c_1 e^{(\lambda_1 - \lambda_{m+1})z} + \cdots + (\lambda_m - \lambda_{m+1})c_m e^{(\lambda_m - \lambda_{m+1})z}$$

for all $z \in \mathbb{C}$. Because the $\lambda_i - \lambda_{m+1}$ are still pairwise distinct, may apply the induction hypothesis to conclude

$$0 = (\lambda_1 - \lambda_{m+1})c_1 = \cdots = (\lambda_m - \lambda_{m+1})c_m.$$

Since $\lambda_i \neq \lambda_{m+1}$ for $1 \leq i \leq m$ gives $c_1 = \cdots = c_m = 0$. Then our above hypothesis is just $c_{m+1}e^{\lambda_{m+1}z} = 0$ for all $z \in \mathbb{C}$, but again using $e^{\lambda_{m+1}z} \neq 0$, we conclude $c_{m+1} = 0$ as well.

Exercise 5: Suppose by way of contradiction that such a function F exists, and define $G(z) := F(z) - \mathrm{Log}(z)$ for all $z \in D'$, where

$$D' := D \setminus \{x \in \mathbb{R}; x < 0\}.$$

Because D' is a domain and $G'(z) = F'(z) - \frac{1}{z} = 0$ for all $z \in D'$, we know that $G(z) = c$ for some $c \in \mathbb{C}$. Thus $F(z) = c + \mathrm{Log}(z)$ for all $z \in D'$, but because F is analytic in D', it is also continuous there, yet

$$\lim_{t \to 0+} F\left(-\frac{3}{2} + it\right) = c + \lim_{t \to 0+} \mathrm{Log}\left(-\frac{3}{2} + it\right) = \log(3/2) + i\pi,$$

$$\lim_{t \to 0-} F\left(-\frac{3}{2} + it\right) = c + \lim_{t \to 0-} \mathrm{Log}\left(-\frac{3}{2} + it\right) = \log(3/2) - i\pi,$$

a contradiction.

Exercise 6:

(1) Just as we approached the nth roots of unity, the same method gives us the roots

$$\omega_k = x^{1/n} e^{2\pi i (k/n)} \tag{0.3}$$

for $k = 0, 1, 2, \ldots, n-1$ where $x^{1/n}$ is the positive nth root of x. To see why there is more than one way to define a root, let's take a brief stroll back to the first time you learned about square roots: Recall, they said that, for a positive number x, the square root of x was the number y having the property that $y^2 = x$. But, already we see an issue. For example, when $x = 4$, $y = 2$ and $y = -2$ both satisfy $y^2 = 4 = x$. For convenience, there was then a definition made (all in the realm of the real number system): Given a positive real number x, the square root of x is the (unique) positive number y such that $y^2 = x$ and it is denoted by $y = \sqrt{x}$. The definition could have just has easily been made the other way, i.e., declaring that the square root of a positive number x is the unique negative number y such that $y^2 = x$. Generalizing this a little further, we see that, for a positive real number x, there are n numbers ω, all given by (0.3), for which $\omega^n = x$ – all are good candidates for a definition for the nth root of x. But of course, for uniformity, we simply choose the real (and positive one), ω_0.

(2) Yes, it is true. To see this, simply observe that, for any $k = 0, 1, \ldots, n-1$,

$$\mathrm{Log}(\omega_k) = \log(x^{1/n}) + i\,\mathrm{Arg}(e^{2\pi(k/n)i})$$

Now, for $k/n \leq 1/2$, $\mathrm{Arg}(e^{2\pi i(k/n)}) = 2\pi k/n$. Thus, for any $k = 1, 2, \ldots, n-1$ for which $k/n \leq 1/2$, we have

$$\mathrm{Log}(\omega_k) = \log(x^{1/n}) + 2\pi(k/n)i$$

and so

$$\begin{aligned} e^{n\,\mathrm{Log}(\omega_k)} &= e^{n\left(\log(x^{1/n})+2\pi(k/n)i\right)} \\ &= e^{n\log(x^{1/n})+2\pi ki} \\ &= e^{\log((x^{1/n})^n)}e^{2\pi ki} \\ &= e^{\log(x)}\cdot 1 \\ &= x \end{aligned}$$

where we have used the fact that k in an integer. When $1/2 < k/n \leq 1$, things are slightly more difficult (but for no interesting reason). For these values, the principal branch of the argument gives

$$\mathrm{Arg}(e^{2\pi(k/n)i}) = 2\pi(k/n) - 2\pi = 2\pi(k-n)/n$$

and therefore (using identical calculations)

$$e^{n\,\mathrm{Log}(\omega_k)} = e^{\log(x)}e^{n(2\pi(k-n)/n)i} = xe^{2\pi(k-n)i} = x\cdot 1 = x.$$

(3) Yes, note that

$$\mathrm{Log}(\omega_0) = \mathrm{Log}(x^{1/n}) + i\,\mathrm{Arg}(e^{0i}) = \frac{1}{n}\mathrm{Log}(x) + i\cdot 0 = \frac{1}{n}\mathrm{Log}(x).$$

So, it makes sense that that is a reasonable definition. Now, given $z \in \mathcal{D}$, i.e., $z \neq 0$, we consider

$$z^{1/n} = e^{\frac{1}{n}\mathrm{Log}(z)}.$$

Observe that

$$(z^{1/n})^n = \left(e^{\frac{1}{n}\mathrm{Log}(z)}\right)^n = e^{\frac{n}{n}\mathrm{Log}(z)} = e^{\mathrm{Log}(z)} = \mathrm{Log}(z)$$

where we have used the fundamental inverse property of the logarithm: $e^{\mathrm{Log}(z)} = z$ for every $z \in \mathcal{D}$.

(4) To see that $z \mapsto z^{1/2}$ is not continuous at -1, it is enough to find a sequence $z_n \to -1$ for which

$$\lim_{n\to\infty} z_n^{1/2} \neq (-1)^{1/2} = i.$$

Note, I have used the fact that $e^{(1/2)\mathrm{Log}(-1)} = e^{(1/2)(0+i\pi)} = e^{i\pi/2} = i$ and so our definition of the square root makes sense. Okay, mimicking the proof I gave in class to show that $\mathrm{Log}(z)$ isn't continuous at $z = -1$, consider the sequence $z_n = e^{i\pi(1+1/n)}$ defined for $n = 1, 2, \ldots$. We note that, by the continuity of the exponential map,

$$\lim_{n\to\infty} e^{i\pi(1+1/n)} = e^{i\pi(1+0)} = e^{i\pi} = -1.$$

We see, however, that $\operatorname{Arg}(z_n) = \pi(1 - 1/n) - 2\pi = -\pi(1 + 1/n)$ and therefore

$$\begin{aligned} z_n^{1/2} = e^{(1/2)\operatorname{Log}(z_n)} &= e^{(1/2)(\operatorname{Log}(|z_n|+i\operatorname{Arg}(z_n))} \\ &= e^{(1/2)(0+i(-\pi(1+1/n)))} \\ &= e^{-i(1+1/n)\pi/2} \end{aligned}$$

and therefore

$$\lim_{n\to\infty} z_n^{1/2} = \lim_{n\to\infty} e^{-i(1+1/n)\pi/2} = e^{-i\pi/2} = -i \neq i = (-1)^{1/2},$$

as was asserted. Thus $z \mapsto z^{1/2}$, defined using the principal branch of the logarithm, is not continuous at -1.

By virtue of Theorem 3 on Page 69 (the composition of differentiable functions is differentiable), $z \mapsto z^{1/2} = e^{(1/2)\operatorname{Log}(z)}$ is differentiable at every point $z \in \mathcal{D}^*$ because $\operatorname{Log}(z)$ is analytic on $\mathcal{D}^*$ and $z \mapsto e^z$ is entire. Consequently, $z \mapsto z^{1/2}$ is analytic on $\mathcal{D}^*$. Since differentiable functions are continuous, we conclude that $z \mapsto z^{1/2}$ must also be continuous on $\mathcal{D}^*$.

We should, perhaps, think about if it is possible for $z^{1/2}$ to be continuous at any point of $\mathbb{C} \setminus \mathcal{D}^*$ (this the branch cut of $\operatorname{Log}(z)$). By cooking up a completely analogous argument (to that for $z = -1$), you will find that $z^{1/2}$ cannot be continuous (and hence not differentiable) at any strictly negative real number. That leaves the final point $z = 0$. Since $z \mapsto z^{1/2} = e^{(1/2)\operatorname{Log}(z)}$ isn't defined at $z = 0$, it doesn't make sense to ask about continuity there and so we are done. This still seems kind of silly though because the "real" square root function is defined and continuous at $x = 0$. If you share my sentiment, you should know that this shortcoming of $z \mapsto z^{1/2}$ can be "fixed" by simply declaring that the function has the value of 0 at $z = 0$ (this is the idea of a removable discontinuity) and the function then does become continuous at $z = 0$ – but we won't worry about that.

(5) Yes, this is not all that hard, we just have to select a branch of the logarithm that is, in fact, continuous on $\operatorname{Re}(z) < 0$. So, to this end, let's define

$$\operatorname{Log}_{3\pi/4}(z) = \log(|z|) + \arg_{3\pi/4}(z)$$

where $\arg_{3\pi/4}$ is the branch of the argument with branch cut along the ray $\{z = re^{-i\pi/4} : r > 0\}$ and range $(-\pi/4, 7\pi/4]$. It's not difficult to work out that

$$\arg_{3\pi/4}(z) = \operatorname{Arg}(e^{-i3\pi/4}z) + \frac{3\pi}{4}.$$

It is straightforward to verify that $\operatorname{Log}_{3\pi/4}$ is analytic except on its branch cut which lies in the forth quadrant. Consequently, this is an analytic function on $\operatorname{Re}(z) < 0$ and thus continuous there. With it we define

$$z^{1/2} = e^{(1/2)\operatorname{Log}_{3\pi/4}(z)}$$

for $z \neq 0$. For any such z, we can write $z = |z|e^{i\theta}$ where $(-\pi/4, 7\pi/4]$, with this representation, we observe that $\operatorname{Arg}_{3\pi/4}(z) = \theta$. Thus,

$$\left(z^{1/2}\right)^2 = e^{(2/2)\operatorname{Log}_{3\pi/4}(z)} = e^{\operatorname{Log}(|z|)}e^{iArg(z)} = |z|e^{i\theta} = z,$$

as desired.

(6) Okay.

(a) By properties of the exponential, we have

$$z^{\alpha+\beta} = e^{(\alpha+\beta)\,\mathrm{Log}(z)} = e^{\alpha\,\mathrm{Log}(z)+\beta\,\mathrm{Log}(z)} = e^{\alpha\,\mathrm{Log}(z)}e^{\beta\,\mathrm{Log}(z)} = z^{\alpha}z^{\beta}.$$

for any $z \in \mathcal{D}^*$.

(b) I hope doing this question got you to think a little bit. In fact, it is not true that $(z^\alpha)^\beta = z^{\alpha\beta}$ for all complex numbers α, β, z. To see this, observe that

$$(e^2\pi i) = e^{2\pi i(\mathrm{Log}(e))} = e^{2\pi i(1+i0)} = e^{2\pi i} = 1$$

(as it must be). And therefore

$$(e^{2\pi i})^{2\pi i} = 1^{2\pi i} = e^{2\pi i\,\mathrm{Log}(1)} = e^{2\pi i(0)} = e^0 = 1.$$

On the other hand,

$$e^{(2\pi i)(2\pi i)} = e^{-4\pi^2} < 1$$

and therefore

$$(e^{2\pi i})^{2\pi i} \neq e^{(2\pi i)(2\pi i)}.$$

The property does, however, hold whenever $\beta \in \mathbb{Z}$ (that I mean to add that as a hypotheses).

(c) By our first Item,

$$z^\alpha z^{-\alpha} = z^0 = 1$$

and so $z^{-\alpha} = 1/z^\alpha$, as desired.

(d) By its definition (and Theorem 3 on Page 69), $z \mapsto z^\beta$ is analytic on $\mathcal{D}^*$ and

$$\frac{d}{dz}z^\beta = \frac{d}{dz}\left(e^{\beta\,\mathrm{Log}(z)}\right)) = e^{\beta\,\mathrm{Log}(z)}\beta\frac{d}{dz}\,\mathrm{Log}(z) = \beta z^\beta \frac{d}{dz}\,\mathrm{Log}(z)$$

for $z \in \mathcal{D}^*$. Now, as we showed in class,

$$\frac{d}{dz}\,\mathrm{Log}(z) = \frac{1}{z} = z^{-1}$$

whenever $z \in \mathcal{D}^*$. Therefore

$$\frac{d}{dz}z^\beta = \beta z^\beta z^{-1} = \beta z^{\beta-1}$$

for all $z \in \mathcal{D}$ where we have used Item 1.