MATH 4220 HOMEWORK 5 SOLUTIONS

Exercise 1: To make things easier, we will first solve the problem if the hinge is at the origin instead of at $1+i$. Let $\arg_0$ be the inverse of the function

$$\exp : \{z \in \mathbb{C}; 0 \leq \mathrm{Im}(z) < 2\pi\} \to \mathbb{C}^*.$$

That is, $\arg_0$ is the branch of the logarithm with branch cut the positive real axis, which takes values in the range $\{z \in \mathbb{C}; 0 \leq \mathrm{Im}(z) < 2\pi\}$. Let

$$f(z) := A \cdot \arg_0(z) + B,$$

where $A, B \in \mathbb{R}$ are to be determined. Along the ray $\mathrm{Arg}(z) = \frac{3\pi}{2}$, we have

$$0 = f(z) = \frac{3\pi}{2}A + B,$$

while along $\mathrm{Arg}(z) = 0$, we have

$$10 = f(z) = B.$$

Thus, we must choose $A = -\frac{20}{3\pi}$ and $B = 10$. Next, we appropriately translate f to obtain the solution $\phi(z) := f(z-1-i)$.

Finally, we compute

$$\phi(0) = f(-1-i) = -\frac{20}{3\pi} \cdot \frac{5\pi}{4} + 10 = \frac{5}{3}.$$

Exercise 2: From the equation

$$z = \cos(w) = \frac{e^{iw} + e^{-iw}}{2} = \frac{1}{2}e^{-iw}(e^{2iw}+1),$$

we get

$$(e^{iw})^2 - 2ze^{iw} + 1 = 0.$$

Appealing to the quadratic formula then gives

$$e^{iw} = z + \sqrt{z^2-1},$$

which is multi-valued. Thus, we get

$$iw = \log\left(z + \sqrt{z^2-1}\right),$$

which is also multi-valued.

First choose a branch of the square root which is analytic near z^2-1, and then choose a branch of the logarithm analytic near $z + \sqrt{z^2-1}$. Then the chain rule gives

$$\begin{aligned} \frac{d}{dz}\cos^{-1}(z) &= \frac{1}{i(z+\sqrt{z^2-1})}\left(1 + \frac{z}{\sqrt{z^2-1}}\right) \\ &= \frac{1}{i\sqrt{z^2-1}}. \end{aligned}$$

Depending on where the branch cut used to define the square root is relative to $z^2 - 1$, we have

$$\sqrt{z^2-1} = \pm i\sqrt{1-z^2}.$$

Note: It is more natural to leave the expression for $\cos^{-1}(z)$ as $\frac{1}{i\sqrt{z^2-1}}$, where the branch of the square root used is the same as in the formula found in the first part of this problem. This is because in general, if $w \mapsto \sqrt{w}$ is a branch of the square root, then $\sqrt{-w}$ will be either $i\sqrt{w}$ or $-i\sqrt{w}$ depending on where w is relative to the branch cut. Thus, in the formula in (11) in section 3.5 of the book, the branch of the square root does not necessarily coincide with the branch chosen in formula (9).

Exercise 3: Solving the equation

$$z = \tan(w) = \frac{1}{i}\frac{e^{2iw}-1}{e^{2iw}+1}$$

for e^{2iw} gives

$$e^{2iw} = \frac{1+iz}{1-iz},$$

so taking the (multi-valued) log of both sides gives

$$w = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right).$$

When $z \neq \pm i$, we can pick a branch of the logarithm which is analytic near $\frac{1+iz}{1-iz}$, $1+iz$, and $1-iz$, and use the chain rule to compute

$$\begin{aligned}\frac{d}{dz}\tan^{-1}(z) &= \frac{i}{2}\frac{d}{dz}\log(1-iz) - \frac{i}{2}\frac{d}{dz}\log(1+iz)\\ &= \frac{i}{2}\left(\frac{-i}{1+iz} + \frac{-i}{1-iz}\right)\\ &= \frac{1}{1+z^2}\end{aligned}$$

Here, we used the fact that $\log(\frac{w_1}{w_2}) - \log(w_1) + \log(w_2) \in 2\pi i\mathbb{Z}$.

Exercise 4: For any $a \in \mathbb{C}$, the function e^{az} is entire by the chain rule, with derivative ae^{az}. Taking $a = \log(c)$ gives

$$\frac{d}{dz}c^z = \log(c)e^{\log(c)z} = \log(c)c^z.$$

Exercise 5: (a) Using the parametrization $\gamma(t) := (1-t)z_1 + tz_2$, $t \in [0,1]$, we note that $\frac{d\gamma}{dt} = z_2 - z_1$, so that the length of this line segment is

$$\int_0^1 \left|\frac{d\gamma}{dt}\right| dt = \int_0^1 |z_2 - z_1| dt = |z_2 - z_1|.$$

(b) Using the parametrization $\gamma(t) := z_0 + re^{2\pi it}$, $t \in [0,1]$, we note that $\frac{d\gamma}{dt} = 2\pi ire^{2\pi it}$, so that the length of this line segment is

$$2\pi \int_0^1 r|e^{2\pi it}|dt = 2\pi \int_0^1 rdt = 2\pi r.$$

Exercise 6:

(1) By the chain rule, we have

$$z_2'(s) = \frac{d}{ds}(z_1(\phi(s))) = z_1'(\phi(s))\phi'(s)$$

for $s \in [c,d]$. We note that, because $\phi'(s) \geq 0$,

$$|z_2'(s)| = |z_1'(\phi(s))\phi'(s)| = |z_1'(\phi(s))|\phi'(s)$$

for all $s \in [c,d]$. Now, using the fact that $\phi(c) = a$ and $\phi(d) = b$, we may write

$$\int_c^d |z_2'(s)|\,ds = \int_{\phi(a)}^{\phi(b)} |z_1'(\phi(s))|\phi'(s)\,ds$$

We may now use "u-substitution" (with $u = \phi(s)$ this is really the change-of-variables formula from any calc 1 book) and find that

$$\int_{\phi(a)}^{\phi(b)} |z_1'(\phi(s))|\phi'(s)\,ds = \int_a^b |z_1'(u)|\,du = \int_a^b |z_1(t)|\,dt$$

from which it follows that

$$\int_c^d |z_2'(s)|\,ds = \int_a^b |z_1'(t)|\,dt.$$

(2) Here,

$$\int_a^b |z_1(t)|\,dt = \int_0^1 |2\pi ie^{2\pi it}|\,dt = \int_0^1 2\pi\,dt = 2\pi$$

and

$$\int_c^d |z_2(s)|\,dt = \int_0^{-1} |-2\pi ie^{-2\pi is}|\,dt = \int_0^{-1} 2\pi\,ds = -2\pi$$

where we have used the property of the Riemann integral that

$$\int_a^b = -\int_b^a.$$

Thus, these integrals are not equal. This statement proven in Item 1 didn't apply here because $\phi'(t) = \frac{d}{dt}(-t) = -1 < 0$.

(3) So, in Item 1 we showed that the length of a curve is preserved under changes of parameterization where both parameterizations have the same direction. That is, any change between admissible parameterizations. This length, however, is not preserved when there is a change in directions, but the way we have defined length. We could, of course, redefine length by putting

$$\ell_{\text{New}}(\gamma) = \left|\int_a^b |z'(t)|\,dt\right|$$

where z is any smooth parameterization (regardless of whether or not it is admissible) and things will work out. Still, as with line integrals, it is important to have a notion of the integral that has a direction associated to it and so we keep our definition as it stands.