Math 4220: Homework 6

This is the sixth homework assignment for Math 4220. I have broken the homework assignment into two parts. Part I has exercises that you should do (and I expect you to do) but you need not turn in. The exercises in the second part of the homework are exercises you should write up and submit via Gradescope. Your solutions should be consistent with the directions in the syllabus. If you get stuck on any part of the homework, please come and see me or Max. More importantly, have fun!

Part I (Do not write up)

Exercise 1. Read Sections 4.4a and 4.5 of the course textbook.

Exercise 2. Do Exercises 1, 3, 5, 9, 11, 13, 15, and 19 in Section 4.4 of the textbook.

Exercise 3. Do Exercises 1, 3, 5, 7, 11, and 13 in Section 4.5 of the textbook.

Part II (Write up and Submit via Gradescope)

Exercise 1. Please do Exercise 2 in Section 4.4 of the course textbook.

Exercise 2. Please do Exercise 16 in Section 4.4 of the course textbook.

Exercise 3. Please do Exercise 18 in Section 4.4 of the course textbook.

Exercise 4. Please do Exercise 8 in Section 4.5 of the course textbook.

Exercise 5. Please do Exercise 14 in Section 4.5 of the course textbook.

In the following exercise, we use Cauchy's theorem to obtain the values of certain integrals that are very difficult to compute without invoking complex analysis. As you will see, this type of idea will become a theme for us. That is, we shall take things that seemingly have nothing to do with complex analysis and, by widening our perspective to the complex plane, we shall use our complex-analysis toolbox to obtain/compute amazing results.

Exercise 6. In the theory of optics (specifically near-field Fresnel diffraction), one commonly encounters the improper integrals

$$C = \int_0^\infty \cos(x^2)\,dx \qquad \text{and} \qquad S = \int_0^\infty \sin(x^2)\,dx. \tag{1}$$

If you recall from calculus, these improper integrals are defined by

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx$$

and are said to converge when the above limit exists. In contrast to the improper integral

$$\int_0^\infty \frac{1}{(1+x)^2}\,dx,$$

which converges because the function $f(x) = (x+1)^{-2}$ becomes very very small as $x \to \infty$ and so the area under the curve is finite, the integrands $\cos(x^2)$ and $\sin(x^2)$ do not become uniformly small as $x \to \infty$. Instead, these functions oscillate very very rapidly and the resultant "signed area" under the curve cancels itself out. For these reason, the integrals in (1) are called oscillatory integrals and their convergence is a delicate matter. In this exercise, you will calculate C and S using Cauchy's theorem.

1. To this end, consider the function $f(z) = e^{-z^2}$ and, for a given $R > 0$, the contour $\Gamma = \gamma_1 + \gamma_2 - \gamma_3$ where γ_1 is the line segment on the real axis from 0 to $z = R$, γ_2 is the portion of the unit circle going from $z = R$ to $z = Re^{i\pi/4}$, and γ_3 is the line segment from $z = 0$ to $z = Re^{i\pi/4}$. The contour Γ is illustrated in Figure 1. Using Cauchy's theorem, argue that

$$\int_{\gamma_3} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

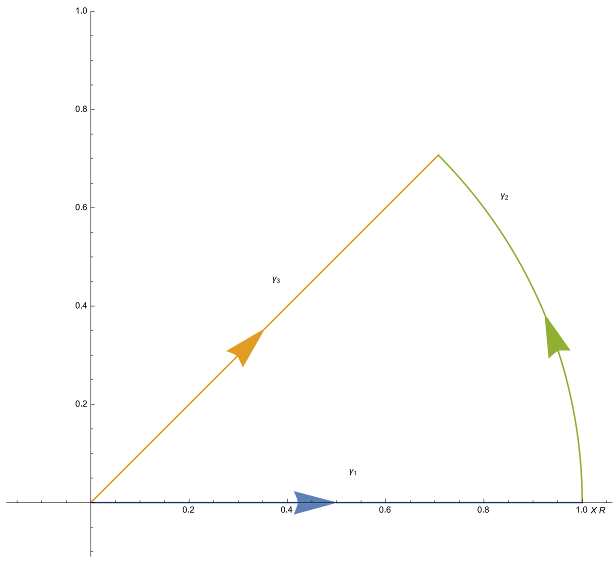

Figure 1: The contour Γ

2. Compute the integrals $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$ and, by simplifying as much as possible, show that

$$\lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz = \frac{\sqrt{\pi}}{2}$$

and

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = 0.$$

Hint: You may use the fact that

$$2\int_0^\infty e^{-x^2}\,dx = \int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}.$$

3. Now, compute

$$\lim_{R\to\infty} \int_{\gamma_3} f(z)\,dz;$$

in doing this, you should find both improper integrals C and S.

4. Finally, by equating both of your computations, you should find that

$$(a+ib)(C+iS) = \frac{\sqrt{\pi}}{2}.$$

for some a and b (that will be known to you from the previous step). Find C and S. Hint: Two complex numbers are equal if and only if their real and imaginary parts are equal.