MATH 4220 HOMEWORK 6 SOLUTIONS

Exercise 1: For $j = 0, 1$, let $z_j(s,t)$ be a deformation of the contour Γ_j to a point w_j, so that $z_j(s,0) = \Gamma_j(s)$, $z_j(s,1) = w_j$ for all $s \in [0,1]$. Because D is a domain, we can find a path $\eta : [0,1] \to D$ with $\eta(0) = w_0$ and $\eta(1) = w_1$. We Then a deformation of Γ_0 to Γ_1 is defined by deforming Γ_0 to w_0 via z_0, then deforming w_0 to w_1 via η, and finally deforming w_1 to Γ_2 via z_1. More concretely, the deformation of Γ_0 to Γ_1 is given as follows:

$$\Gamma(s,t) := \begin{cases} \gamma_0(s, 3t) & t \in [0, 1/3] \\ \eta(3t-1) & t \in [1/3, 2/3] \\ \gamma_1(s, 3-3t) & t \in [2/3, 1] \end{cases} .$$

Note that $\Gamma(s,t)$ is continuous because it is obtained by "patching together" several continuous functions which agree where their domains overlap.

Exercise 2: Using the standard parametrization $\gamma(t) := e^{it}$, $t \in [0, 2\pi]$, we compute

$$\int_{|z|=1} \frac{1}{z^k} dz = \int_0^{2\pi} \frac{1}{e^{ikt}} i e^{it} dt = i \int_0^{2\pi} e^{i(1-k)t} dt.$$

If $k = 1$, the integrand on the right is uniformly 1, so we get $\int_{|z|=1} z^{-1} dz = 2\pi i$. If $k \neq 1$, we get

$$\int_{|z|=1} \frac{1}{z^k} dz = \frac{1}{1-k}(e^{2\pi(1-k)i} - 1) = 0.$$

By the linearity of (line) integrals, we conclude

$$\int_{|z|=1} f(z) dz = 2\pi i A_1 + \int_{|z|=1} g(z) dz,$$

but $\int_{|z|=1} g(z) dz = 0$ since g is analytic in $\{|z| \leq 1\}$.

Exercise 3:
(a) First observe that $z \mapsto \frac{1}{z^2(z-1)^3}$ is analytic everywhere except $z = 0$ and $z = 1$. Moreover, we can deform $\{|z| = 2\}$ to $\{|z| = R\}$ without passing through the points $z = 0$, $z = 1$, so this claim follows from the Deformation Invariance Theorem (Theorem 8 in section 4.4).

(b) When $|z| = R \geq 2$, we can estimate $|z - 1| \geq R - 1 > 0$, so that

$$|I(R)| \leq \int_{|z|=R} \left| \frac{1}{z^2(z-1)^3} \right| \cdot |dz| \leq \int_{|z|=R} \frac{1}{R^2(R-1)^3} |dz| = \frac{2\pi}{R(R-1)^3}.$$

(c) Because $\lim_{R\to\infty} R(R-1)^3 = +\infty$, this follows from (b).

(d) By part (a), we know $I = I(R)$ for every $R > 2$, so the claim follows from part (c).

Exercise 4: We choose the standard parametrization $\gamma(t) := z_0 + re^{it}$, $t \in [0, 2\pi]$, and apply Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt.$$

By differentiating Cauchy's integral formula n times (or equivalently, using Theorem 19 of section 4.5), we also have

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}dz = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{r^{n+1}e^{i(n+1)t}} ire^{it}dt \\ &= \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{it})e^{-int}dt. \end{aligned}$$

Exercise 5: (a) In this case, Γ can be deformed to a circle centered at z without passing through the point ζ, so the Deformation Invariance Theorem gives $G(z) = \cos(z)$, and the continuity of cos implies that the requested limit in (a) is equal to $\cos(2 + 3i)$.

(b) In this case, the function $\zeta \mapsto \frac{\cos(\zeta)}{\zeta - z}$ is analytic on a domain containing the region bounded by Γ, so $G(z) = 0$, and the requested limit is 0.

Exercise 6: (1) Because f is entire, this follows from the Deformation Invariance Theorem.

(2) Because

$$\int_{\gamma_1} f(z)dz = \int_0^R e^{-t^2}dt,$$

we can take $R \to \infty$ to get

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)dz = \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

Next, we use

$$\int_{\gamma_2} f(z)dz = \int_0^{\frac{\pi}{4}} e^{-R^2e^{2it}} iRe^{it}dt$$

to estimate

$$\left|\int_{\gamma_2} f(z)dz\right| \le R\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2t)}dt.$$

Next, we observe that because $\cos|[0, \frac{\pi}{4}]$ is concave ($\cos'' = -\cos < 0$ on this interval), we have $\cos(t) \ge \frac{4}{\pi}(\frac{\pi}{4} - t)$, so

$$\left|\int_{\gamma_2} f(z)dz\right| \le Re^{-R^2}\int_0^{\frac{\pi}{4}} e^{R^2\frac{4}{\pi}t}dt = Re^{-R^2}\left.\frac{\pi}{4R^2}e^{R^2\frac{4}{\pi}t}\right|_{t=0}^{t=\frac{\pi}{4}} \le \frac{\pi}{4R}.$$

In particular, $\lim_{R\to\infty}\int_{\gamma_2} f(z)dz = 0$.

(3) Since $e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}}$ and $e^{\frac{\pi i}{2}} = i$, we can compute

$$\int_{\gamma_3} f(z)dz = \left(\frac{1+i}{\sqrt{2}}\right)\int_0^R e^{-it^2}dt = \left(\frac{1+i}{\sqrt{2}}\right)\left(\int_0^R \cos(t^2)dt - i\int_0^R \sin(t^2)dt\right).$$

Taking $R\to\infty$ gives

$$\lim_{R\to\infty}\int_{\gamma_3} f(z)dz = \left(\frac{1+i}{\sqrt{2}}\right)(C - iS).$$

(4) Combining parts $(1)-(3)$ gives

$$\left(\frac{1+i}{\sqrt{2}}\right)(C - iS) = \lim_{R\to\infty}\int_{\gamma_3} f(z)dz = \frac{\sqrt{\pi}}{2}.$$

Rearranging this expression gives

$$(C+S) + i(C-S) = \sqrt{\frac{\pi}{2}},$$

so $C = S$, and $C = \sqrt{\frac{\pi}{8}} = S$.