MATH 4220 HOMEWORK 7 SOLUTIONS

Exercise 1: Note that $g(z) := \frac{f(z)}{3z^2}$ is analytic on a neighborhood of the annulus $\{1 \le |z| \le 2\}$, and $|g(z)| = \frac{1}{3}|f(z)| \le 1$ on $\{|z| = 1\}$, while $|g(z)| = \frac{1}{12}|f(z)| \le 1$ on $\{|z| = 2\}$. Thus the maximum modulus principle gives $|g(z)| \le 1$, or equivalently, $|f(z)| \le 3|z|^2$, whenever $1 \le |z| \le 2$.

Exercise 2: Because $h(z) := f(z)/g(z)$ is analytic on D and continuous on $D \cup B$, we can apply the maximum modulus principle to get

$$\frac{|f(z)|}{|g(z)|} = |h(z)| \le \max_{\zeta \in B} |h(\zeta)| = \max_{\zeta \in B} \frac{|f(\zeta)|}{|g(\zeta)|} \le 1$$

for all $z \in D \cup B$.

Exercise 3: Write $P(z) = \lambda(z - a_1) \cdot (z - a_m)$. Then

$$\frac{P'(z)}{P(z)} = \frac{1}{(z - a_1) \cdot (z - a_n)} \sum_{j=1}^{m} (z - a_1) \cdots (z - a_{j-1})(z - a_{j+1}) \cdots (z - a_m)$$
$$= \sum_{j=1}^{m} \frac{1}{z - a_j}.$$

By reindexing, we can assume that $a_1, ..., a_\ell$ lie inside Γ, while $a_{\ell+1}, ..., a_m$ lie outside. Then dividing both sides of the above equation by $2\pi i$ and integrating over Γ gives (by Cauchy's integral formula)

$$\frac{1}{2\pi i} \int_\Gamma \frac{P'(z)}{P(z)} dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - a_j} dz = \sum_{j=1}^{\ell} 1 = \ell.$$

Exercise 4: The Poisson integral formula gives

$$\phi(0) = \frac{\rho^2}{2\pi} \int_0^{2\pi} \frac{\phi(\rho e^{it})}{\rho^2} dt = \frac{1}{2\pi} \int_0^{2\pi} \phi(\rho e^{it}) dt.$$

Multiplying this equation by ρ and integrating gives

$$\frac{1}{2} R^2 \phi(0) = \int_0^R \phi(0) \rho d\rho = \frac{1}{2\pi} \int_0^{2\pi} \int_0^R \phi(\rho e^{it}) \rho d\rho dt = \frac{1}{2\pi} \int_D \phi(z) dx dy.$$

Exercise 5: First observe that

$$R^2 + r^2 - 2rR\cos(t - \theta) \le R^2 + r^2 + 2rR = (R + r)^2,$$
$$R^2 + r^2 - 2rR\cos(t - \theta) \ge R^2 + r^2 - 2rR = (R - r)^2,$$

so we can use the Poisson integral formula and the previous exercise to estimate

$$\phi(re^{i\theta}) \leq \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi(Re^{it})}{(R-r)^2} dt = \frac{R^2 - r^2}{(R-r)^2}\phi(0) = \frac{R+r}{R-r}\phi(0),$$
$$\phi(re^{i\theta}) \geq \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi(Re^{it})}{(R+r)^2} dt = \frac{R^2 - r^2}{(R+r)^2}\phi(0) = \frac{R-r}{R+r}\phi(0).$$

Exercise 6: (a) Because Γ_R is a simple closed countour containing z, this follows from Cauchy's integral formula.

(b) $\text{Im}(\overline{z}) = -\text{Im}(z) < 0$, so z is not contained in the simple closed countour Γ_R.

(c) We compute

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\Gamma_R} f(\zeta) \left(\frac{z - \overline{z}}{(\zeta - z)(\zeta - \overline{z})} \right) d\zeta \\ &= \frac{1}{2\pi i} \int_{-R}^{R} f(\zeta) \frac{2i\text{Im}(z)}{(t-z)(t-\overline{z})} dt + \int_{C_R^+} f(\zeta) \frac{2i\text{Im}(z)}{(\zeta - z)(\zeta - \overline{z})} d\zeta. \end{aligned}$$

In the first integrand, we note that since $\overline{t} = t$, we have

$$(t - z)(t - \overline{z}) = (t - z)\overline{(t - z)} = |t - z|^2.$$

(d) For any $\zeta \in C_R^+$, we can estimate

$$|\zeta - z| \geq |\zeta| - |z| = R - |z|,$$

and similarly $|\zeta - \overline{z}| \geq R - |z|$. Combining this with $|f(z)| \leq K$, we can estimate

$$\left| f(\zeta) \frac{2i\text{Im}(z)}{(\zeta - z)(\zeta - \overline{z})} \right| \leq \frac{2|\text{Im}(z)| \cdot |f(\zeta)|}{(R - |z|)^2} \leq \frac{2K|\text{Im}(z)|}{(R - |z|)^2}.$$

Because the length of C_R^+ is πR, we can use $|f(z)| \leq K$ to estimate

$$\left| \frac{1}{2\pi i} \int_{C_R^+} f(\zeta) \frac{2i\text{Im}(z)}{(\zeta - z)(\zeta - \overline{z})} d\zeta \right| \leq \frac{1}{2\pi} \pi R \cdot \frac{2K|\text{Im}(z)|}{(R - |z|)^2}.$$

Note that $\text{Im}(z) \geq 0$, so $\text{Im}(z) = |\text{Im}(z)|$.

(e) Because (using the change of variables $t = \xi - x$)

$$\int_{-R}^{R} \left| \frac{\phi(\xi, 0)}{(\xi - x)^2 + y^2} \right| d\xi \leq \frac{K}{y^2} \int_{-R-x}^{R-x} \frac{1}{(t/y)^2 + 1} dt \leq \frac{K\pi}{y} < \infty,$$

we know that the term in the absolute value is absolutely integrable. We can thus take $R \to \infty$ in part (c) (using part (d) to eliminate the other integral)

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\zeta) \frac{2\text{Im}(z)}{|t - z|^2} dt = \frac{\text{Im}(z)}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\zeta)}{|t - z|^2} dt + i\frac{\text{Im}(z)}{\pi} \int_{-\infty}^{\infty} \frac{\psi(\zeta)}{|t - z|^2} dt.$$

The claim follows by taking real parts of both sides, and writing $z = x + iy$, noting that $|t - z|^2 = (t - x)^2 + y^2$.