MATH 4220 HOMEWORK 8 SOLUTIONS

Exercise 1: (a) If $S_n := \sum_{j=0}^n c_j$, then $\overline{S}_n = \sum_{j=0}^n \overline{c_j}$. By the continuity of complex conjugation, we have

$$\overline{S} = \overline{\lim_{n\to\infty} S_n} = \lim_{n\to\infty} \overline{S_n} = \lim_{j\to 0}^{n} \sum_{j=0}^{n} \overline{c_j}.$$

$(b), (c)$ The usual limit rules give

$$\lambda S = \lambda \lim_{j\to\infty} S_n = \lim_{j\to\infty} \lambda S_n = \lim_{n\to\infty} \sum_{j=0}^{n} \lambda c_j.$$

(c) If $T_n := \sum_{j=0}^n d_j$, then

$$S + T = \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n = \lim_{n\to\infty} (S_n + T_n) = \lim_{n\to\infty} \sum_{j=0}^{n} (c_j + d_j).$$

Exercise 2: (i) Note that $\sum_{j=2}^N \frac{1}{j^p}$ is the area under the step function which equals $1/j^p$ on $[j-1, j)$ for each integer $j \geq 2$. Note that this function lies below $1/x^p$; in particular, we can estimate

$$\int_1^N \frac{1}{x^p} dx = \sum_{j=2}^{N} \int_{j-1}^{j} \frac{1}{x^p} dx \geq \sum_{j=2}^{N} \frac{1}{j^p}$$

. When $p > 1$, we conclude that

$$\int_1^N \frac{1}{x^p} dx = \frac{1}{p-1} \left(1 - \frac{1}{N^{p-1}}\right).$$

In particular, the comparison test gives

$$\sum_{j=1}^{N} \frac{1}{j^p} \leq 1 + \frac{1}{p-1} < \infty.$$

$(ii)(a)$ Because $\frac{1}{j(j+1)} < \frac{1}{j^2}$, the claim follows from the comparison test and part (i) (taking $p = 2$).

$(ii)(b)$ Because $|\sin(k^2)| \leq 1$, this follows from the comparison test and part (i) (taking $p = 3/2$).

$(ii)(c)$ We estimate

$$\left| \frac{k^2 i^k}{k^4 + 1} \right| \leq \frac{k^2}{k^4} = \frac{1}{k^2},$$

so the claim follows from the comparison test and part (i) (taking $p = 2$).

(*d*) When $k \geq 2$, we have $k^3 - 1 \geq k^3 - \frac{1}{2}k^3 = \frac{1}{2}k^3$, so we can estimate

$$\left| (-1)^k \left(\frac{5k+8}{k^3 - 1} \right) \right| \leq \frac{13k}{\frac{1}{2}k^3} = \frac{26}{k^2}.$$

The claim then follows from the comparison test part (i) (taking $p = 2$).

Exercise 3: We need to find $\epsilon > 0$ such that, for any $N \in \mathbb{N}$, there exist $n \geq N$ and $|z| < 1$ such that

$$\left| \frac{1}{1-z} - \sum_{j=0}^{n} z^j \right| > \epsilon.$$

We observe that

$$\frac{1}{1-z} - \sum_{j=0}^{n} z^j = \frac{1}{1-z} - \frac{1 - z^{n+1}}{1-z} = \frac{z^{n+1}}{1-z}.$$

In particular, fixing N, and taking $z = 1 - \delta$ gives

$$\left| \frac{1}{1-z} - \sum_{j=0}^{n} z^j \right| = \delta^{-1} (1 - \delta)^{n+1}.$$

By taking $\delta > 0$ sufficiently small (depending on N), we can ensure that the right hand side is at least 1.

Exercise 4: We can write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \left(1 - \frac{(z - z_0)}{(\zeta - z_0)} \right)^{-1}.$$

For any $\alpha \in (0, 1)$, we have uniformly convergence of the geometric series

$$\left(1 - \frac{(z - z_0)}{(\zeta - z_0)} \right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^j$$

uniformly in the set $|z - z_0| < \alpha |\zeta - z_0|$. The claim then follows from part (b) of Exercise 1.

Alternatively, set $f(z) := \frac{1}{\zeta - z}$. Suppose by induction that $f^{(n)}(z) = \frac{n!}{(}\zeta - z)^{n+1}$. Then

$$f^{(n+1)}(z) = (n+1) \frac{n!}{(\zeta - z)^{n+2}} = \frac{(n+1)!}{(\zeta - z)^{n+2}},$$

so we have established this formula for all $n \in \mathbb{N}$. The formal Taylor series for f based at the point z_0 is thus

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

To check that this converges, we can factor out $(\zeta - z)^{-1}$ and view the Taylor series as a geometric series as in the previous paragraph.

Exercise 5: (i) For any $z \in D$, Theorem 3 gives

$$f(z) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = 0.$$

(ii) For any $z \in D$, Theorem 3 gives

$$\begin{aligned} f(-z) &= \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!}(-z)^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(2k)}(0)}{(2k)!}((-z)^2)^k \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(2k)}(z_0)}{(2k)!} z^{2k} = f(z). \end{aligned}$$

Exercise 6: $(i)(a)$ For any $|z| \leq 1$, we can estimate

$$\begin{aligned} \left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| &= \lim_{m\to\infty} \left| \sum_{k=n+1}^{m} \frac{z^k}{k!} \right| \leq \limsup_{m\to\infty} \sum_{k=n+1}^{m} \frac{|z|^k}{k!} \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \frac{1}{(n+1)!}\left(1 + \sum_{k=n+2}^{\infty} \frac{(n+1)!}{k!}\right). \end{aligned}$$

For any $k \geq n+2$, we have

$$\frac{(n+1)!}{k!} = \frac{1}{(n+2)(n+3)\cdots(k-1)k} \leq \frac{1}{(n+2)^{k-n-1}},$$

so we can estimate

$$\sum_{k=n+2}^{\infty} \frac{(n+1)!}{k!} \leq \sum_{k=n+2}^{\infty} \frac{1}{(n+2)^{k-n-1}} = \sum_{k=1}^{\infty} \frac{1}{(n+2)^k} = \frac{(n+2)^{-1}}{1-(n+2)^{-1}} = \frac{1}{n+1}.$$

$(i)(b)$ Arguing as above gives

$$\left| \sin(z) - \sum_{k=0}^{n} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \right| \leq \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{(2n+3)!}\left(1 + \sum_{k=n+2}^{\infty} \frac{1}{(2n+4)\cdots(2k+1)}\right).$$

Thus

$$\begin{aligned} \sum_{k=n+2}^{\infty} \frac{1}{(2n+4)\cdots(2k+1)} &\leq \sum_{k=n+2}^{\infty} \frac{1}{(2n+4)^{k-(n+1)}(2n+5)^{k-(n+1)}} \\ &= \sum_{k=1}^{\infty} \frac{1}{(2n+4)^k(2n+5)^k} \\ &= \frac{(2n+4)^{-1}(2n+5)^{-1}}{1-(2n+4)^{-1}(2n+5)^{-1}} = \frac{1}{4n^2+16n+19}. \end{aligned}$$

Combining expressions gives the claim.

(ii) The problem asks us to find the smallest integer n such that

$$f(n) := \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1}\right) \leq 10^{-5}.$$

Note f is decreasing in n, and that $1 + \frac{1}{n+1}$, so we will need n to satisfy $(n+1)! \geq 10^{-5}$. The smallest such n is $n = 8$, with $9! = 362880$. We note that

$$f(8) = \frac{1}{362880} \cdot \frac{10}{9} = 326592 < 10^{-5},$$

so we need the first 9 terms of the expansion for e^z to obtain the desired error bound.