Math 4220: Homework 9

This is the ninth homework assignment for Math 4220. I have broken the homework assignment into two parts. Part I has exercises that you should do (and I expect you to do) but you need not turn in. The exercises in the second part of the homework are exercises you should write up and submit via Gradescope. Your solutions should be consistent with the directions in the syllabus. If you get stuck on any part of the homework, please come and see me or Max. More importantly, have fun!

Part I (Do not write up)

Exercise 1. Read Sections 5.6 and 5.7 of the course textbook. Note: We will cover only 5.6 in class, but both are very straightforward and so you should read them carefully.

Exercise 2. Do Exercises 1, 3, 5, 7, 9, 13, and 17 in Section 5.6 of the textbook.

Exercise 3. Do Exercises 1, 3, 5, and 7 in Section 5.7 of the textbook.

Part II (Write up and Submit via Gradescope)

A glimpse at Fourier Series by way of Laurent series

Before I present the first two exercises, the following is a brief account of what we covered in lecture (minus the material on the heat equation and the history lesson – both of which can be found in the wonderful book **Fourier Analysis** by T. Körner): As we did in class, suppose that, for some $0 \leq r < 1 < R$, $F(z)$ is analytic on the annulus

$$A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}.$$

We note that $A_{r,R}$ contains the unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$ which we take with positive orientation. In this case, the Laurent series theorem says that

$$F(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

for $z \in A_{r,R}$ where, for each $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i}\int_S \frac{F(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

Let's parameterize S by $\zeta(t) = e^{it}$ for $-\pi \leq t \leq \pi$ and observe that, for each $n \in \mathbb{Z}$,

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_S \frac{F(\zeta)}{\zeta^{n+1}}\,d\zeta \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{F(e^{it})}{(e^{it})^{n+1}}(ie^{it})\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})e^{-int}\,dt \end{aligned}$$

Define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(t) = F(e^{it})$$

for $t \in \mathbb{R}$ and observe that $f(t)$ is necessarily periodic of period 2π. With this, we observe that

$$a_n = \widehat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

for each $n \in \mathbb{Z}$ and hence the Laurent series identity for $F(z)$ can be written equivalently by

$$F(z) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) z^n$$

which is valid for $z \in A_{r,R}$. If we are only interested in reproducing the values of F on S, since $S \in A_{r,R}$, we obtain (by setting $z = e^{i\theta}$),

$$f(\theta) = F(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta}$$

for all $\theta \in \mathbb{R}$. We can state this as a theorem.

Theorem 1. *For $0 \le r < 1 < R$, suppose that F is analytic on the annulus $A_{r,R} = \{z \in \mathbb{Z} : r < |z| < R\}$ and consider the 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(t) = F(e^{it})$ for $t \in \mathbb{R}$. Then*

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{in\theta}$$

for $\theta \in \mathbb{R}$ where, for each $n \in \mathbb{Z}$,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt.$$

In the preceding theorem, the sequence of numbers $\{\widehat{f}(n)\}$ are known as the Fourier coefficients of f and the process of obtaining them is often referred to as Fourier decomposition. Armed with these Fourier coefficients $\{\widehat{f}(n)\}$, the series

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta}$$

is called the Fourier series of f. For a given f, forming this series is often called “Fourier synthesis”. Though the 2π-periodic function f was gotten by evaluating the analytic function F on S, there was really nothing else in the theorem that speaks explicitly of F. With this observation, suppose that I hand you a 2π-periodic function f (which may or may not have some analytic F in the background) and assume that f is sufficiently nice so that you can integrate it and compute its Fourier coefficients $\widehat{f}(n)$. With these coefficients, you can then write down the Fourier series for f. We ask:

1. Does the Fourier series converge? Precisely, if for $N \in \mathbb{N}$, we define

   $$S_N(\theta) = \sum_{n=-N}^{N} \widehat{f}(n)e^{in\theta}$$

   for $\theta \in \mathbb{R}$ (called the Nth Fourier polynomial/partial sum), does the limit

   $$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta} = \lim_{N\to\infty} S_N(\theta)$$

   exist? If so, for which θ's?

2. In the case that it does converge, when do we have

   $$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta}?$$

Exercise 1. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(\theta) = \cos(2\theta)$ for $\theta \in \mathbb{R}$.

1. Compute the Fourier coefficients $\widehat{f}(n)$ of f.

2. Using these coefficients, compute the Fourier polynomials $S_N(\theta)$ for $N \in \mathbb{N}$ and simplify as much as possible.

3. Plot f along with S_N for $N = 1, 2, 3, \dots 10$ for $-3\pi \le \theta \le 3\pi$. Of course, it's possible that many of these polynomials are the same (for different N) and, in that case, you only need to plot one. Please indicate this, however.

4. By studying your Fourier polynomials and your plot from the previous item, answer the following:

(a) Does the Fourier series converge? If so, for what values of θ?

(b) For those values of θ for which is converges, does it converge to f?

5. Is there an analytic function $F(z)$ for which $f(\theta) = f(e^{it})$ for $t \in \mathbb{R}$? If so, what is it? If not, explain.

Exercise 2. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(\theta) = \frac{1}{\pi}(\theta - m\pi)$ whenever $(m-1)\pi < \theta \leq (m+1)\pi$ where m is an even integer. This is often called the sawtooth function.

1. Compute the Fourier coefficients $\widehat{f}(n)$ of f.

2. Using these coefficients, compute the Fourier polynomials $S_N(\theta)$ for $N \in \mathbb{N}$ and simplify as much as possible.

3. Plot f along with S_N for $N = 1, 2, 3, \ldots 10$ for $-3\pi \leq \theta \leq 3\pi$.

4. By studying your Fourier polynomials and your plot from the previous item, answer the following:

   (a) Does the Fourier series converge? If so, for what values of θ?

   (b) For those values of θ for which is converges, does it converge to f?

5. Is there an analytic function $F(z)$ for which $f(\theta) = f(e^{it})$ for $t \in \mathbb{R}$? If so, what is it? If not, explain.

Exercise 3. Please do Exercise 2 in Section 5.6 of the course textbook.

Exercise 4. Please do Exercise 14 in Section 5.6 of the course textbook.

Exercise 5. Please do Exercise 2 in Section 5.7 of the course textbook.

Exercise 6. Please do Exercise 6 of Section 5.7 of the course textbook.