MATH 4220 HOMEWORK 8 SOLUTIONS

Exercise 1: (1) We compute

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\theta)e^{in\theta}d\theta = \frac{1}{4\pi}\int_{-\pi}^{\pi}(e^{2i\theta}+e^{-2i\theta})e^{in\theta}d\theta$$
$$= \begin{cases} \frac{1}{2} & n = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

(2) We compute $S_0 = S_1(\theta) = 0$, whereas for all $N \geq 2$, we have

$$S_N(\theta) = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \cos(2\theta).$$

(4) The Fourier series converges to f since $S_N = f$ for all $N \geq 2$.

(5) The analytic function $F(z) := \frac{1}{2}(z^2+z^{-2})$ satisfies $F(e^{i\theta}) = \cos(2\theta)$.

Exercise 2: (1) When $n \neq 0$, integration by parts gives

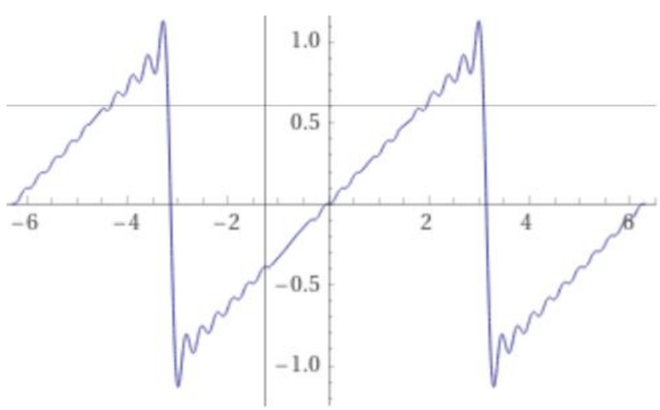
$$\widehat{f}(n) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\theta e^{-in\theta}d\theta = \frac{1}{2\pi^2}\left(\frac{\theta}{-in}e^{-in\theta}\Big|_{\theta=-\pi}^{\theta=\pi} + \int_{-\pi}^{\pi}\frac{1}{in}e^{-in\theta}d\theta\right)$$
$$= \frac{i}{2\pi^2 n}\left(\pi e^{-in\pi}+\pi e^{in\pi}\right) = \frac{i}{\pi n}\cos(n\pi) = \frac{(-1)^n i}{\pi n}.$$

Moreover, because $\theta \mapsto \theta/\pi$ is odd, we know $\widehat{f}(0) = 0$.

(2) We may thus compute

$$S_N(\theta) = \frac{i}{\pi}\sum_{n=1}^{N}(-1)^n\left(\frac{1}{n}e^{in\theta}+\frac{1}{-n}e^{-in\theta}\right) = -\frac{2}{\pi}\sum_{n=1}^{N}\frac{(-1)^n}{n}\sin(n\theta).$$

(3) The following is the plot of S_{10} from -2π to 2π:



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(4) The previous plot suggests that S_N should converge to f away from the discontinuities $\theta = (2k+1)\pi$, $k \in \mathbb{Z}$. However, we can see directly that $S_N((2k+1)\pi) = 0$, so while $\lim_{N\to\infty} S_N((2k+1)\pi) = 0$ exists, it is not equal to $f(0) = 1$.

(5) Such a function F cannot exist because this would imply $f(\theta) = F(e^{i\theta})$ would be continuous, which it is not.

Exercise 3: Observe that the function

$$g(z) := 2\cos(z) - 2 + z^2$$

has everywhere convergent Taylor expansion given by

$$g(z) = -2 + z^2 + 2\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 2\sum_{k=2}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!},$$

so has a zero of order 4 at $z = 0$. This means that $f(z) = 1/(g(z)^2)$ has a pole of order 8 at $z = 0$ by Lemma 8 of section 5.6.

Exercise 4: By replacing $f(z)$ with $f(z - z_0)$, we can assume $z_0 = 0$. Suppose f does not achieve values arbitrarily close to some $c \in \mathbb{C}$ near $z = 0$. This means there exists $\delta > 0$ and some puncture disk $\mathbb{D}(0, r_0) \setminus \{0\}$ such that $|f(z) - c| \geq \delta$ for all $z \in \mathbb{D}(0, r_0) \setminus \{0\}$. Then

$$g(z) := \frac{1}{f(z) - c}$$

is an analytic function satisfying $|g(z)| \leq \delta^{-1}$ for all $z \in \mathbb{D}(0, r_0)$. By Problem 13, f has a removable singularity at $z = 0$; that is,

$$w := \lim_{z\to 0} g(z) \in \mathbb{C}$$

exists. Because g is nonvanishing on $\mathbb{D}(0, r_0) \setminus \{0\}$, it is in particular not identically zero, so we can write $g(z) = z^k h(z)$ for some $k \in \mathbb{N}$ and some analytic function h on $\mathbb{D}(0, r_0)$ with $h(0) \neq 0$. Thus

$$z^k f(z) = z^k c + \frac{1}{h(z)},$$

where the right hand side is uniformly bounded. In particular, Problem 13 implies that $f(z)$ has a removable singularity, or a pole of order at most k.

Problem 5: By definition, $g(z) := f(1/z)$ has a removable singularity at ∞, so extends to an analytic function on some disk $\mathbb{D}(0, 2r_0)$. The Taylor series

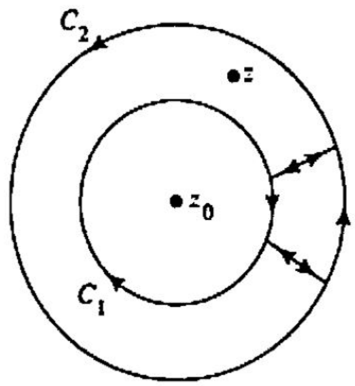
$$\sum_{n=0}^{\infty} a_n z^n.$$

of g then converges uniformly to g on $\mathbb{D}(0, r_0) \setminus \{0\}$. This is equivalent to the uniform convergence

$$\sum_{n=0}^{\infty} a_n z^{-n} = f(z)$$

on $\mathbb{D}(0, r_0^{-1})$.

Problem 6: We consider a contour similar to that used in the proof of Theorem 14 in section 5.5 of the textbook:



(c)

where the outer contour is $\partial\mathbb{D}(0, R)$ and the inner contour is $-\Gamma$ (Γ is negatively oriented). Because $g(z) = f(1/z)$ is analytic and $g(0) = 0$, we can write $g(z) = zh(z)$ for some analytic function h in the interior of $\{1/z; z \in \Gamma\}$. Setting $\gamma(t) := Re^{it}$, $t \in [0, 2\pi]$ for $R >> 1$, we can compute

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}(0,R)} \frac{\frac{1}{\zeta}h(\frac{1}{\zeta})}{\zeta - z} d\zeta\right| &= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{R^{-1}e^{-it}h(R^{-1}e^{-it})}{Re^{it} - z} iRe^{it} dt\right| \\ &\leq \frac{\max_{\overline{\mathbb{D}}(0,R^{-1})} |h|}{2\pi} \int_0^{2\pi} \frac{1}{R - |z|} dt \\ &\leq \frac{2}{R} \max_{\overline{\mathbb{D}}(0,R^{-1})} |h|.\end{aligned}$$

On the other hand, Cauchy's integral formula applied to the above contour gives

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}(0,R)} \frac{\frac{1}{\zeta}h(\frac{1}{\zeta})}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

so the claim follows by taking $R \to \infty$.