Math 4220: Pre-Prelim Solutions

Part II (Also, do not turn in)

Exercise 1. Let Γ be the square with vertices $z = 0, 1, 1+i$ and i traversed in the counterclockwise direction.

1. Compute $\int_\Gamma z\,dz$
2. Compute $\int_\Gamma \overline{z}\,dz$.

Solution 1. We shall consider the parameterization $z : [0,1] \to \mathbb{C}$ of Γ given by

$$z(t) = \begin{cases} 4t + 0i & 0 \le t \le 1/4 \\ 1 + 4(t-1/4)i & 1/4 \le t \le 1/2 \\ (1 - 4(t-1/2) + i & 1/2 \le t \le 3/4 \\ 0 + (1 - 4(t-3/4))i & 3/4 \le t \le 1. \end{cases}$$

It should be noted that I could have first broken Γ into four pieces, each of which is a smooth curve and then parameterized the pieces individually (and we would have obtained parameterizations that align with the "pieces" of the above piece-wise function). In the end, it all works out the same.

1. Using the above parameterization, we find

$$\begin{aligned} \int_\Gamma z\,dz &= \int_0^{1/4} z(t)z'(t)\,dt + \int_{1/4}^{1/2} z(t)z'(t)\,dt + \int_{1/2}^{3/4} z(t)z'(t)\,dt + \int_{3/4}^{1} z(t)z'(t)\,dt \\ &= \int_0^{1/4} (4t)(4)\,dt + \int_{1/4}^{1/2} (1+4(t-1/4)i)(4i)\,dt \\ &\qquad + \int_{1/2}^{3/4} ((1-4(t-1/2)) + i)(-4)\,dt + \int_{3/4}^{1} (0 + (1-4(t-3/4))i)(-4i)\,dt \\ &= \frac{1}{2} + \left(i - \frac{1}{2}\right) + \left(\left(-1 + \frac{1}{2}\right) - i\right) + \frac{1}{2} \\ &= 0. \end{aligned}$$

   Knowing what we know from Section 4.3, this result is not surprising, for Γ is a closed loop and $f(z) = z$ has an antiderivative $F(z) = z^2/2$.

2. We have

$$\begin{aligned} \int_\Gamma \overline{z}\,dz &= \int_0^{1/4} \overline{z(t)}z'(t)\,dt + \int_{1/4}^{1/2} \overline{z(t)}z'(t)\,dt + \int_{1/2}^{3/4} \overline{z(t)}z'(t)\,dt + \int_{3/4}^{1} \overline{z(t)}z'(t)\,dt \\ &= \int_0^{1/4} \overline{(4t)}(4)\,dt + \int_{1/4}^{1/2} \overline{(1+4(t-1/4)i)}(4i)\,dt \\ &\qquad + \int_{1/2}^{3/4} \overline{((1-4(t-1/2)) + i)}(-4)\,dt + \int_{3/4}^{1} \overline{(0 + (1-4(t-3/4))i)}(-4i)\,dt \\ &= \frac{1}{2} + \left(i + \frac{1}{2}\right) + \left(\left(-1 + \frac{1}{2}\right) + i\right) - \frac{1}{2} \\ &= 2i. \end{aligned}$$

   By what we know from Section 4.3, $\overline{z}$ cannot have an antiderivative on any open set G containing Γ.

Exercise 2. As we saw in class,

$$\int_C z^n\,dz = \begin{cases} 0 & n \ne -1 \\ 2\pi i & n = -1. \end{cases}$$

where C is the unite circle, centered at 0 and directed/oriented counterclockwise.

1. Using the above fact and what you know about the properties of the contour integral, show that

$$\int_C P(z)\,dz = 0$$

for any polynomial $P(z)$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function and suppose that, for each $\epsilon > 0$, there exists a polynomial $P(z) = P_\epsilon(z)$ such that

$$|f(z) - P(z)| < \epsilon \tag{1}$$

for all $z \in C$. Show that

$$\int_C f(z)\,dz = 0.$$

Hint: Use Theorem 5. Note: The property (1) says that, within any desired accuracy, you can find a polynomial $P(z)$ which approximates $f(z)$ to that accuracy uniformly on C. This property is akin to saying that, given any real number x, there is a rational number as close as you desire to x (think truncated decimal expansions). This idea begs the question: Which functions f have this property? As it turns out, this is an very important question in mathematics there are many theorems which address it in one context or another. One such theorem is called the Weierstrass approximation theorem, a theorem which has deep applications in mathematics and its (ahem) applications.

3. It is no surprise that e^z has the property discussed in the previous item (though we haven't yet seen it, this is because e^z is best expressed as Taylor series, i.e., a limit of polynomials). By what you just proved, it must be true that

$$\int_C e^z\,dz = 0.$$

Please confirm this by directly computing this integral. Note: If you're using the parameterization $z(t) = e^{it}$ for $t \in [2\pi]$, It is helpful to note things like

$$\frac{d}{dt} e^{\cos(t)} \cos(\sin(t)) = -e^{\cos(t)} \cos(\sin(t)) \sin(t) - e^{\cos(t)} \sin(\sin(t)) \cos(t).$$

Solution 2.

1. Using linearity of the integral and the given fact, we have

$$\begin{aligned}
\int_C P(z)\,dz &= \int_C (a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)\,dz \\
&= a_n \int_C z^n\,dz + a_{n-1} \int_C z^{n-1}\,dz + \cdots a_1 \int_C z\,dz + a_0 \int_C z^0\,dz \\
&= a_n \cdot 0 + a_{n-1} \cdot 0 + \cdots + a_1 \cdot 0 + a_0 \cdot 0 \\
&= 0.
\end{aligned}$$

2. Given any $\epsilon > 0$, select $P = P_\epsilon$ for which (1) holds and observe that

$$\left|\int_C f(z)\,dz\right| = \left|\int_C f(z)\,dz - 0\right| = \left|\int_C f(z)\,dz - \int_C P(z)\,dz\right| = \left|\int_C (f(z) - P(z))\,dz\right| \le \epsilon \ell(C) = \epsilon 2\pi$$

where we have used the fact that $|f(z) - P(z)| < \epsilon$ and Theorem 5 on Page 170. Thus, we have shown that, for all $\epsilon > 0$,

$$\left|\int_C f(z)\,dz\right| \le 2\pi\epsilon.$$

If the integral $\int_C f(z)\,dz = w \neq 0$, choosing $\epsilon = |w|/4\pi$ would yield the inequality $|w| \le |w|/2$ which is nonsense. Thus the integral must be zero.

3. This is a straightforward (though slightly difficult) computation. If you need help checking it, let me know.

Exercise 3. For this exercise, please don't rely on anything in Section 4.3.

1. Please do Exercise 16 in Section 4.2.

2. Please do Exercise 17 in Section 4.2.

Solution 3. 1. Let $z : [0,1] \to \mathbb{C}$ be a parameterization of the directed arc γ. In particular, $z(t)$ is once continuously differentiable and has $z(0) = \alpha$ and $z(1) = \beta$. By Theorems 3 and 4,

$$\int_\gamma z\,dz = \int_0^1 z(t)z'(t)\,dt = \frac{z(1)^2}{2} - \frac{z(0)^2}{2} = \frac{\beta^2}{2} - \frac{\alpha^2}{2}$$

where we have used the fact that $F(t) = z(t)^2/2$ has

$$F'(t) = z(t)z'(t).$$

2. Since any closed contour γ can be seen as having the same initial and terminal point, i.e., $\alpha = \beta$, we obtain

$$\int_\gamma z\,dz = 0.$$

Exercise 4. Please do Exercise 6 in Section 4.3.

Solution 4. Using Theorem 6 (noting that $\mathrm{Log}(z - z_0$ is analytic everywhere beyond the illustrated branch cut, we obtain

$$\int_{C_{\alpha,\beta}} \frac{dz}{z - z_0}\,dz = \mathrm{Log}(\beta - z_0) - \mathrm{Log}(\alpha - z_0)$$

where $C_{\alpha,\beta}$ denotes the solid portion of the circle drawn in Figure 4.25. By letting α and β approach τ in the way indicated, we see that

$$\lim_{\alpha\to\tau} \mathrm{Re}(\mathrm{Log}(z + z_0)) = \lim_{\beta\to\tau} \mathrm{Re}(\mathrm{Log}(z + z_0)) = \mathrm{Re}(\mathrm{Log}(\tau + z_0) = \mathrm{Log}\,|\tau + z_0|$$

and therefore

$$\int_C \frac{dz}{z - z_0} = \lim_{\alpha,\beta\to\tau} \int_{C_{\alpha,\beta}} \frac{dz}{z - z_0}\,dz = \mathrm{Log}(\beta - z_0) - \mathrm{Log}(\alpha - z_0) = \lim_{\beta\to\tau} \mathrm{Arg}(\beta - z_0)i - \lim_{\alpha\to\tau} \mathrm{Arg}(\alpha - z_0)i$$

where, of course, Arg denotes the principal argument. Doing this in the way indicated, we know that the values of the principal argument differ by exactly 2π. Thus

$$\int_C \frac{dz}{z - z_0} = 2\pi i.$$

Exercise 5. Please do Exercise 12 in Section 4.3.

Solution 5. Let γ denote the line segment from z_1 to z_2, which is necessarily inside the unit disk (provided that z_1 and z_2 are). Upon noting that $\ell(\gamma) = |z_2 - z_1|$, an application of Theorems 5 (on Page 170) and Theorem 6 on (173) yields

$$|f(z_2) - f(z_2)| = \left|\int_\gamma f'(z)\,dz\right| \le M \cdot \ell(\gamma) = M|z_2 - z_1|,$$

as desired.