Math 4220: Pre-Prelim Problems

This is the Pre-Prelim problem set, the homework before our first prelim for MA4220. As with our usual homework assignments, I have broken this up into two parts and, also as usual, Part I has exercises that you should do (and I expect you to do) but you need not turn in. Part II contains exercises that are slightly more involved (as usual); however, in contrast to our normal homework assignments) I will not be collecting solutions. You should still treat this homework seriously and come to office hours (either Max's or mine) when you have questions. I will post the solutions sometime next week before the prelim. As always, have fun!

Part I (Do not turn in)

Exercise 1. Read Sections 4.2 and 4.3 of the course textbook.

Exercise 2. Do Exercises 1, 3, 5, 6, 7, 9, 12, 13, and 14 Section 4.2 of the textbook.

Exercise 3. Do Exercises 1, 2, 3, 4, 5, 7, 8, 9, 10 in Section 4.3 of the textbook.

Part II (Also, do not turn in)

Exercise 1. Let Γ be the square with vertices $z = 0, 1, 1+i$ and i traversed in the counterclockwise direction.

1. Compute $\int_\Gamma z\,dz$
2. Compute $\int_\Gamma \overline{z}\,dz$.

Exercise 2. As we saw in class,

$$\int_C z^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases}$$

where C is the unite circle, centered at 0 and directed/oriented counterclockwise.

1. Using the above fact and what you know about the properties of the contour integral, show that

   $$\int_C P(z)\,dz = 0$$

   for any polynomial $P(z)$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function and suppose that, for each $\epsilon > 0$, there exists a polynomial $P(z) = P_\epsilon(z)$ such that

   $$|f(z) - P(z)| < \epsilon \tag{1}$$

   for all $z \in C$. Show that

   $$\int_C f(z)\,dz = 0.$$

   Hint: Use Theorem 5. Note: The property (1) says that, within any desired accuracy, you can find a polynomial $P(z)$ which approximates $f(z)$ to that accuracy uniformly on C. This property is akin to saying that, given any real number x, there is a rational number as close as you desire to x (think truncated decimal expansions). This idea begs the question: Which functions f have this property? As it turns out, this is an very important question in mathematics there are many theorems which address it in one context or another. One such theorem is called the Weierstrass approximation theorem, a theorem which has deep applications in mathematics and its (ahem) applications.

3. It is no surprise that e^z has the property discussed in the previous item (though we haven't yet seen it, this is because e^z is best expressed as Taylor series, i.e., a limit of polynomials). By what you just proved, it must be true that

   $$\int_C e^z\,dz = 0.$$

   Please confirm this by directly computing this integral. Note: If you're using the parameterization $z(t) = e^{it}$ for $t \in [2\pi]$, It is helpful to note things like

   $$\frac{d}{dt}e^{\cos(t)}\cos(\sin(t)) = -e^{\cos(t)}\cos(\sin(t))\sin(t) - e^{\cos(t)}\sin(\sin(t))\cos(t).$$

Exercise 3. For this exercise, please don't rely on anything in Section 4.3.

1. Please do Exercise 16 in Section 4.2.

2. Please do Exercise 17 in Section 4.2.

Note: All preceding exercises had you conclude that $\int_\Gamma f(z) = 0$ for various closed curves Γ and (nice) functions $f(z)$. Can you think about which were analytic (and where) or were not analytic?

Exercise 4. Please do Exercise 6 in Section 4.3.

Exercise 5. Please do Exercise 12 in Section 4.3.