The Bar Spectral Sequence

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1 The Bar Spectral Sequence

The study of iterated loop spaces is a central aspect of homotopy theory; nowadays the study of infinite loop spaces goes by the name of *stable homotopy theory* and accounts for a sizeable portion of modern homotopy theoretic research. In the study of such spaces, it is natural to think about *deloopings*: given a loop space T, a delooping of T is a space BT such that $T \simeq \Omega BT$. For example, for any (discrete or topological) group G, the delooping BG is also called the *classifying space* of G and has the property that isomorphism classes of G-bundles on a space X are in natural bijection with homotopy classes $[X, BG]$.

Much of the study of infinite loop spaces has been concentrated on *recognition principles*: given a space T, how can we tell whether or not a space BT satisfying the defining property $T \simeq \Omega BT$ exists? One first observation is that loop spaces possess a multiplication via loop concatenation, so a necessary condition is for T to possess a multiplication. A further note is that loop concatenation is unital and associative (up to homotopy), and furthermore that every loop has an inverse loop (up to homotopy). It turns out that the "up to homotopy" nature of unitality/associativity is simple in the sense that one may as well assume the multiplication is strictly associative/unital, and this turns out to be necessary and sufficient:

Theorem 1.1. *A space T has a delooping BT if and only if it possesses (up to weak homotopy equivalence) an associative, unital multiplication such that the induced monoid structure on $\pi_0(T)$ is a group.*

In light of this theorem, we will study the category TopAssMon of associative monoids in Top. The elements of this category are spaces T equipped with a unital, associative multiplication. This includes spaces for which $\pi_0(T)$ may not be a group (for instance, $\mathbb{N}$); we include these in our study despite the fact that they cannot possess deloopings for reasons that we will remark upon later.

One major difficulty in the study of loop spaces is the difficulty of understanding their (co)homology: the most obvious example is the (co)homology of Eilenberg-Maclane spaces. By beginning with an abelian group A and repeatedly applying B to obtain $A, BA, B^2A, ...$ we obtain the sequence of Eilenberg-Maclane spaces $K(A, 0), K(A, 1), K(A, 2), ...$ (this follows from the definition of B and the long exact sequence of a fibration applied to $\Omega K(A, n) \to PK(A, n) \to K(A, n)$). The (co)homology of these spaces is both deeply interesting and highly nontrivial: in fact, the cohomology of $K(A, n)$ is exactly the group of cohomology operations for cohomology with coefficients in A, an application of the Brown Representability Theorem and the Yoneda Lemma. These groups are complex enough that (to the author's knowledge) there is no place in which they are concisely written down (though descriptions of their localization at a prime p can be found in notes from the 1954-1955 Cartan Seminar). The *Bar Spectral Sequence* is a device designed precisely to deal with this hurdle: Given a topological associative monoid T, the bar spectral sequence takes as input homological data about T and converges to the homology of the delooping BT. Constructing and understanding this spectral sequence is the goal of this section. But first, if we are to have any hope of computing anything about BX, we had better give some sort of explicit construction.

1.1 The Bar Construction

Recall that the space of *Moore loops* of a space X, which we will denote $\Omega^s X$, consists of functions $\ell : [0, n] \to X$ with $\ell(0) = \ell(n) = \text{base}(X)$. It turns out that the natural reparameterization map $\Omega^s X \to \Omega X$ is a homotopy equivalence. Furthermore, $\Omega^s X$ has a strictly unital, associative multiplication via concatenation (the "s" superscript is for *strict*), and hence provides a functor Top $\to$ TopAssMon. The basic idea of the bar construction is this: we want to construct a functor $B : \mathrm{TopAssMon} \to \mathrm{Top}$, with the property that it is almost a one-sided inverse to Ω^s: for a *connected* space X, we should have a weak homotopy equivalence $X \simeq B\Omega^s X$.[1]

First, let us consider the easiest case: the case of an ordinary discrete group. If a discrete group G is (up to homotopy) homotopy equivalent to $\Omega^s X$ for some X, it follows that X is a $K(G,1)$. So in this case, constructing BG is in fact equivalent to constructing a $K(G,1)$. To make things easier on ourselves, we attempt to build BG as a simplicial set: since BG is connected, we may as well take $BG_0 = \{\bullet\}$. Then, every 1-simplex of BG will be a loop, so we might as well take BG_1 to be the set $|G|$. Of course, we can't stop here: if we did, then we would end up with a $K(\mathbb{F}\langle |G| \rangle, 1)$. We need to add in some two simplices to encode the multiplication in G. So for triples of elements $a, b, c \in G$ such that $ab = c$, we throw in a 2-simplex $\Delta_{ab=c}$ such that $\delta_0(\Delta_{ab=c}) = b$, $\delta_1(\Delta_{ab=c}) = c$, and $\delta_2(\Delta_{ab=c}) = b$. This correctly provides a homotopy between loops $ab \sim c$. We do this for every such triple, obtaining $BG_2 = \{(a,b,c) \in G \times G \times G | ab = c\}$. Of course, this is a rather redundant way to write this: since we start with the multiplication of G as given knowledge, if we are given a and b then there is exactly one triple (a,b,c) which appears in this set, the triple (a,b,ab). So we can equivalently write $BG_2 = G \times G$. In this case our boundary maps become written as $d_0(a,b) = b$, $d_1(a,b) = ab$, $d_2(a,b) = a$. Now, what about BG_3? We may have it be the case that we have some π_2 that we need to kill off after having added in these relations, which sounds complicated. But, at this point maybe we see the pattern forming and guess $BG_3 = G \times G \times G$, with the boundary maps given by $d_0(a,b,c) = (b,c)$, $d_1(a,b,c) = (ab,c)$, $d_2(a,b,c) = (a,bc)$, and $d_3(a,b,c) = (a,b)$. If we continue on in this pattern, we create a simplicial set that looks like:

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times G \times G \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times G \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \;\substack{\longrightarrow\\ \longrightarrow}\; *$$

It turns out that this is correct:

Theorem 1.2. *For a discrete group G, denote by BG the geometric realization of the simplicial set with n-simplices G^n, boundary maps described by*

$$d_i(g_1, ..., g_n) = \begin{cases} (g_1, ..., g_n) & \text{if } i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & \text{if } 0 < i < n \\ (g_1, ..., g_{n-1}) & \text{if } i = n \end{cases}$$

and degeneracy maps described by $s_i(g_1, ..., g_n) = (g_1, ..., e, g_i, ..., g_n)$. Then the map $G \to \Omega BG$ given by mapping $a \in G$ to the 1-simplex labelled by a is a homotopy equivalence. In particular, BG is a $K(G,1)$.

Proof. As we will later prove this for the more general case of G a space, we quickly sketch an alternate proof here that works only when G is discrete.

The description above is equivalent to saying that BG is the geometric realization of the nerve of the groupoid with a single object $\bullet$ and $\mathrm{End}(\bullet) = G$. The nerve is a 2-coskeletal connected Kan complex, and so in particular is a $K(\pi, 1)$ where π can be described as the group generated by the 1-simplices, modulo the relations given by the 2-simplices. This is precisely saying that $\pi \cong G$. □

We now turn our attention to the general case: suppose we are given some object $T \in \mathrm{TopAssMon}$. Then we can imitate the above construction, and form the simplicial diagram

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; T \times T \times T \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; T \times T \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; T \;\substack{\longrightarrow\\ \longrightarrow}\; *$$

[1] A little reflection should convince yourself this is the best we could ask for, as the space $\Omega^s X$ depends only on the connected component of the basepoint.

Of course, now T^n is a space, rather than merely a set. Fortunately, this poses no real problem: we can take the geometric realization of a simplicial space with little more difficulty that taking the geometric realization of a simplicial set.[2] Thus, we can define BT to be this geometric realization.

Theorem 1.3. *The above defines a functor B : TopAssMon $\to$ Top. Furthermore, for any connected space X there is a weak homotopy equivalence $X \to B\Omega^s X$.*

Before diving into the proof, we need to develop a little more theory.

1.1.1 The two sided bar construction

We now dive into a little more generality, an enjoyable leap that will clarify which structures are essential to our current activity and which are merely coincidental. Let C be a monoidal category with unit I. Let T be a monoid in T: that is, an object equipped with a a unit morphism $\iota : I \to T$ and a unital, associative multiplication morphism $\mu : T \otimes T \to T$.

Now suppose that in addition to the monoid T, we have objects M and N of C which are modules over T: that is to say, we have maps $M \otimes T \to M$ and $T \otimes N \to N$ which satisfy some identities[3], making M a right T-module and N a left T-module. Then, we can form a simplicial object which we will call $B(M, T, N)$, where $B(M, T, N)_i = M \otimes T^{\otimes i} \otimes N$.

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; M \otimes T \otimes T \otimes N \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; M \otimes T \otimes N \;\substack{\longrightarrow\\\longrightarrow}\; M \otimes N$$

In many cases we also have a functor $|-| : C^{\Delta^{op}} \to C$ called a "geometric realization," which tells us how to take a simplicial object in C and produce an ordinary object of C. For instance, we can take the geometric realization of a simplicial space (a simplicial object in Top, and get a space).

So what's the point of this? Well, for now we won't actually need the full generality, but it helps us state the following:

Lemma 1.4. *Let $T \in$ TopAssMon, and $Y \in$ Top be a module over T. Then $|B(T, T, Y)|$ is homotopy equivalent to Y.*

Proof. We define a function $f : Y \to |B(T, T, Y)|$ via the inclusion of Y into the diagram. In the other direction, we define a function $g : |B(T, T, Y)| \to Y$ via $(t_0, ..., t_n, y) \to (t_0(t_1(\cdots(t_n \cdot y))))$. We claim these are homotopy inverse to each other: $g \circ f$ is easy as it is a genuine inverse. For the other direction, one can write out an explicit simplicial homotopy. □

In particular, the trivial space $*$ is a module over every monoid, and so we get

Corollary 1.5. *$B(T, T, *)$ is weakly contractible, for any $T \in$ TopAssMon.*

The bar construction has some other nice structure: if M is an (S, T) bimodule; that is, if we have compatible actions $S \otimes M \to M$ and $M \otimes T \to M$, then we get an induced action of S on $B(M, T, N)$ by acting levelwise. In particular, T is a (T, T) bimodule, and so we have an action of T on $|B(T, T, N)|$. We may quotient out by this action, and when we do, what we are left with, we obtain precisely $|B(*, T, N)|$. Furthermore, the quotient map has fiber T over the base point. If T was a topological group, it would follow

[2] Exercise to the reader who has not seen this concept before: formulate a definition of geometric realization of a simplicial set $S_\bullet$ as a colimit of spaces $S_n \times \Delta^n$, and then realize that this makes sense even when S_n is a space rather than a set.

[3] The reader is highly encouraged to write out what precisely these identities should be, guided by the simplicial identites applied to the simplicial object we are about to construct.

that the quotient map had fiber T over every point and was in fact a fibre bundle. In the general case this is not necessarily true, but we can prove the next best thing: that the quotient map is a fibration with fibre of constant homotopy type T. First, a lemma:

Lemma 1.6. *Let $T \in AM$ be such that $\pi_0(T)$ is a group. Then the action of T on itself by $\phi_a(t) = at$ is a homotopy equivalence for every $s \in T$.*

Proof. By assumption, for any $a \in T$ there is some element b such that as in π_0, $[ab] = [e]$. In other words, there is a path $p : I \to T$ with $p(0) = e$ and $p(1) = ab$. We define an explicit homotopy between ϕ_e and $\phi_{ab} = \phi_a \circ \phi_b$ by $f(t, i) = \phi_{p(i)}(t)$. This demonstrates that ϕ_b is a right homotopy inverse for ϕ_a, and similarly we can show it is a left homotopy inverse. □

With this in hand, we are able to prove the last lemma we need to prove the bar construction provides a delooping:

Lemma 1.7. *Let $T \in TopAssMon$ be such that $\pi_0(T)$ is a group. Then the sequence $T \to B(T, T, *) \to B(*, T, *)$ is a fibre sequence.*

Proof. The proof can be found in Appendix D of [2]; I have no improvement to offer over the exposition there. □

Putting this all together with a little lemma, we obtain the proof we have been seeking:

Proof that bar construction gives delooping. Consider the map $B(T, T, *) \to B(*, T, *)$. By the above, this is a fibration with fibre T. Inspecting the LES in homotopy of the fibration, and then appealing to our observation about contractibility, we get an explicit WHE $\Omega BT \to T$. For the other direction, we have that $\Omega B \Omega^s X \simeq \Omega^s X \simeq \Omega X$, and so provided X is connected we get that $B\Omega^s X \simeq X$. □

1.2 Algebraic bar construction, constructing the spectral sequence

Having proved that $BT := B(*, T, *)$ is a delooping of T for T a grouplike monoid, we would like to get some handle on the homology of this space. Luckily, it comes with a rather nice filtration, and so we get a spectral sequence with terms on the E^1 page given by $H_i(BT_j, BT_{j-1})$, converging to $H_*(BT)$. Of course, this is not a terribly useful description - we would like to find a more succinct description of the initial page. We will leave this motivation aside for a moment as we redo some of our work in an algebraic context, before bringing it all back together to finish off our description of the spectral sequence. Alas, proving the theorems of this section is an exercise in homological algebra, and thus a work of precision and patience. We sketch all of the main points, and leave the reader interested in verifications to read the excellent exposition in [4] if they feel anything substantial is missing.

As we said before, we can peform bar constructions in any monoidal category. Consider the category of nonnegatively graded chain complexes over a field k; it is equipped with a tensor product of chain complexes defined by

$$(C \otimes D)_n = \oplus_{i+j=n} C_i \otimes D_j$$

This makes it into a monoidal category, and so in particular we can speak of monoids in it and their modules, and in particular bar constructions.

Example Let $T \in$ TopAssMon. Working over a field k, the map $T \times T \to T$ yields a map $C_\bullet(T \times T) \to C_\bullet(T)$, and along with the Kunneth isomorphism $C_\bullet(T) \otimes C_\bullet(T) \to C_\bullet(T \times T)$, this gives $C_\bullet(T)$ the structure of a monoid in Ch_k. Similarly, for M a T-module, $C_\bullet(M)$ becomes a $C_\bullet(T)$-module.

Let Γ be a monoid in Ch_k, N a left Γ-module. Denote by $\tilde{\Gamma}$ the cokernel of the unit map $k[0] \to \Gamma$. We consider the bar construction $B(\Gamma, \tilde{\Gamma}, N)$.

$$\cdots \;\Rrightarrow\; \Gamma \otimes \tilde{\Gamma} \otimes \tilde{\Gamma} \otimes N \;\Rrightarrow\; \Gamma \otimes \tilde{\Gamma} \otimes N \;\rightrightarrows\; \Gamma \otimes N$$

Here we are using the fact that $\tilde{\Gamma}$ is a monoid, and Γ and N are $\tilde{\Gamma}$-modules. We pull a standard trick: we take the simplicial diagram and leverage the additive structure in our category (our ability to add morphisms together) in order to produce a chain complex:

$$\cdots \xrightarrow{d_0-d_1+d_2-d_3} \Gamma \otimes \tilde{\Gamma} \otimes \tilde{\Gamma} \otimes N \xrightarrow{d_0-d_1+d_2} \Gamma \otimes \tilde{\Gamma} \otimes N \xrightarrow{d_0-d_1} \Gamma \otimes N$$

This gives us a chain complex of chain complexes. We can extend this to the right with the structure map of N as a Γ-module to get

$$\cdots \xrightarrow{d_0-d_1+d_2-d_3} \Gamma \otimes \tilde{\Gamma} \otimes \tilde{\Gamma} \otimes N \xrightarrow{d_0-d_1+d_2} \Gamma \otimes \tilde{\Gamma} \otimes N \xrightarrow{d_0-d_1} \Gamma \otimes N \longrightarrow N \longrightarrow 0$$

Theorem 1.8. *The above sequence is exact, and furthermore the terms to the left of N are all projective as Γ-modules. So, this gives a projective resolution of N as a Γ-module.*

Admittedly, we haven't said carefully what we mean by a projective resolution (it's a fair bit more complex than in the discrete case), but the reader interested in the details of these things should seek them out in [UGSS]. Now suppose that we have a right Γ-module M; if we tensor (taking the tensor product over Γ) on the left with M, we will obtain

$$\cdots \xrightarrow{d_0-d_1+d_2-d_3} M \otimes \tilde{\Gamma} \otimes \tilde{\Gamma} \otimes N \xrightarrow{d_0-d_1+d_2} M \otimes \tilde{\Gamma} \otimes N \xrightarrow{d_0-d_1} M \otimes N$$

Which is precisely the definition of $B(M, \Gamma, N)$.

With these big "chain complexes of chain complexes," we call the differentials of the individual chain complexes $\Gamma \otimes \cdots \otimes N$ the *internal differentials* and denote them using δ. We call the differentials $\sum_i (-1)^i d_i$ the *external differentials* and denote them using D. We can form a single chain complex called the *total complex* of this double complex, which we denote $|B(\Gamma, \tilde{\Gamma}, N)|$ by setting

$$|B(\Gamma, \tilde{\Gamma}, N)|_n = \sum_{i+j=n} (\Gamma \otimes \tilde{\Gamma}^{\otimes i} \otimes N)_j$$

And by setting the differential to be $D + (-1)^i \delta$.

Definition Let Γ be a monoid in Ch_k, M and N right and left Γ modules, respectively. $\mathrm{Tor}(M, N)$ is the homology of $B(M, \Gamma, N)$. This inherits a bigrading from the double chain complex structure in its definition.

Now where's the spectral sequence in all of this?

1.3 The spectral sequence

There is sort of an evident filtration on the total complex that we constructed earlier: simply filter by the external degree, let F_0 be the total complex of $M \otimes N$, F_1 be the total complex of $M \otimes \tilde{\Gamma} \otimes N \to M \otimes N$, etc. And, well, where there is a filtration, there is a spectral sequence:

Definition Let Γ be a monoid in Ch_k, M and N be right and left Γ modules, respectively. The *algebraic bar spectral sequence* is the spectral sequence associated to the above filtration of the total complex of $B(M, \tilde{\Gamma}, N)$, converging to $\mathrm{Tor}^{\Gamma}(M, N)$.

To make this spectral sequence useful, we now identify the E^2 page:

Theorem 1.9. *The algebraic spectral sequence associated to $Tor^{\Gamma}(M, N)$ has $E^2_{ij} \cong Tor^{H(\Gamma)}_{ij}(H(M), H(N))$.*

Proof. The E^1 page we can see is given by

$$E^1_{ij} \cong H_j(F^i/F^{i-1}) \cong H_j(M \otimes_{\Gamma} P^i)$$

Equivalently, we have

$$E^1_{i,\bullet} \cong H(M) \otimes_{H(\Gamma)} H(P^i)$$

Which gives us the desired E^2 term, as this is precisely describing the terms of the bar construction $B(H(M), H(\Gamma), H(N))$. $\square$

Theorem 1.10 (The Bar Spectral Sequence)**.** *Let $T \in TopAssMon$. Then there is a spectral sequence with $E^2 \cong Tor^{H(T;k)}(k, k)$ converging to $H_*(BT; k)$.*

Proof. Having already constructed the algebraic spectral sequence, we need only connect the homological algebra and the topology with the observation that $C_*(BT) \cong B(k, H(T; k), k)$. This follows from our simplicial construction of BT. $\square$

1.4 Applications

In this section we denote by $\Lambda_k(t)$ the exterior algebra over the field k on an element t, and by $\Gamma_k[t]$ the divided polynomial algebra over k on an element t.

Theorem 1.11. *Let X be an associative monoid, with $H_*(X) \cong \Lambda_k(t)$. Then $H^*(X) \cong k[u]$.*

Proof. We calculate $\mathrm{Tor}^{\Lambda_k[t]}(k, k)$ where t of degree m. Calculating the terms in the bar construction is fairly straightforward, as $\tilde{\Gamma}$ is one dimensional, concentrated in degree m. Overall, the complex looks like:

$$\begin{array}{c|ccc}
2m & 0 & 0 & [t|t] \\
m & 0 & [t] & 0 \\
0 & [] & 0 & 0 \\
\hline
 & 0 & 1 & 2
\end{array}$$

The differentials are all trivial for degree reasons, so this becomes the E^2 page for our spectral sequence. Note also that for degree reasons, all future differentials are trivial, so this gives us the E^∞ page. The coalgebra structure here is that of the divided polynomial ring (this divided polynomial structure traces its way back to the bar construction having terms which themselves are summations of many terms), so $H_*(BX) \cong \Gamma_k[s]$ for s in degree $m+1$. We get that $H^*(BX)$ is simply the dual of this, as all elements lie in even dimensions. The dual is precisely $k[s]$, as desired. $\square$

The knowledge of how a delooping of a space must behave relative to the original space gives us a useful tool for determining that a space cannot possess a delooping, and therefore places bounds on the amount of algebraic structure a space can support:

Corollary 1.12. *S^7 admits no associative multiplication.*

Proof. Suppose to the contrary that S^7 admits an associative multiplication; then we can deloop it to get the classifying space $B(S^7)$. Note that $H_*(S^7; C_3) = \Lambda_{C_3}[x_7]$, the exterior algebra on a generator in dimension 7. By the previous theorem, $H^*(B(S^7); C_3) = C_3[u_8]$, a polynomial algebra on a generator in dimension 8. Applying the power operations in mod P cohomology, $P = 3$, we have that $P_4(u_8) = u_8^3 \neq 0$, but also $P_4(u_8) = P_1 P_3(u_8) = 0$ (for degree reasons). Thus the delooping cannot exist. $\square$

1.5 Group Completions

We noted earlier that the bar constructions applied to spaces T which could not be delooped because $\pi_0(T)$ was not a group. How are we in this case to interpret the bar construction $B(*, T, *)$? It turns out that in this case, the bar construction provides a *group completion* of T: recall that for a discrete monoid M, a group completion of M is a group A along with a map of monoids $M \to A$ which is initial among maps from M to a group. More succinctly, the group completion functor Mon $\to$ Grp is the left adjoint of the inclusion functor Grp $\to$ Mon. For a topological monoid, we could define group completion similarly, but because inverses may exist only up to homotopy, formulating the universal property is more difficult and requires ∞-categories. A homological definition has been given:

Definition Let $T \in$ TopAssMon. A map $T \to G$ is called a *group completion* if on homology it exhibits the ring localization map

$$H_*(T) \to H_*(T)[\pi_0(T)^{-1}] \cong H_*(G)$$

A theorem of McDuff and Segal says that the map $M \to \Omega BM$ is a group completion (see [2] for an exposition).

One of the ways in which group completions arise while studying loop spaces is that it is often more natural or easier to define monoidal structures than grouplike monoidal structures. For instance, the theory of operads can be used to define monoidal structures, but lacks the ability to define groups - there is no operad such that groups are precisely algebras over that operad. But, via group completion, we can pass from algebras over operads to loop spaces:

Theorem 1.13. *Denote by $\mathbb{F}_{E_k}$ the free E_k-algebra functor, where E_k is the little cubes operad. Then there is a natural transformation $\mathbb{F}_{E_k} \to \Omega^n\Sigma^n$ such that for all X, $\mathbb{F}_{E_k}X \to \Omega^n\Sigma^n X$ is a group completion.*

Furthermore, this commutes with the inclusions $E_k \to E_{k+1}$ and $\Omega^n\Sigma^n \to \Omega^{n+1}\Sigma^{n+1}$. If we take the particular example $X = S^0$, and use the fact that $\mathbb{F}_{E_k} \cong \coprod_i \mathrm{Conf}(i, \mathbb{R}^k)$, the unordered configuration space of i points in $\mathbb{R}^n$, we get a diagram

$$
\begin{array}{ccc}
\coprod_i \mathrm{Conf}(i,\mathbb{R}) & \longrightarrow & \Omega^1 S^1 \\
\downarrow & & \downarrow \\
\coprod_i \mathrm{Conf}(i,\mathbb{R}^2) & \longrightarrow & \Omega^2 S^2 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\coprod_i \mathrm{Conf}(i,\mathbb{R}^\infty) & \longrightarrow & \Omega^\infty S^\infty
\end{array}
$$

The horizontal arrows are group completions. The first row is actually (up to homotopy equivalence) the group completion $\mathbb{N} \to \mathbb{Z}$. The second and final rows we can also give explicit descriptions for: $\mathrm{Conf}(i, \mathbb{R}^2)$ is actually a classifying space for the ith braid group, so the left element in the second row is the disjoint union of classifying spaces of braid groups Br_i. The monoidal structure comes from the operation of placing a braid with k strands next to a braid with ℓ strands, which gives a map $\mathrm{Br}_k \times \mathrm{Br}_\ell \to \mathrm{Br}_{k+\ell}$. Note that this space is what is referred to as a 1-type: π_0 and π_1 are its only nonzero homotopy groups, which follows directly from the fact that $B\mathrm{Br}_i$ is a $K(\mathrm{Br}_i, 1)$. By contrast, $\Omega^2 S^2$ has no trivial homotopy groups, so in this case the group completion drastically alters the homotopy type of the space. This row also offers up some insight on the homotopy groups of S^2: in particular,the map on π_1 is surjective, and the Hopf fibration is mapped to by an element which represents the twisting of a single strand around itself - a commutation relation.[4]

The content of the last row generally goes by the name of the *Barratt-Priddy-Quillen Theorem*. The space $\mathrm{Conf}(i, \mathbb{R}^\infty)$ is a classifying space for Σ_i, the ith symmetric group, so the left space is in some sense an entirely combinatorial object (here the monoidal structure comes from the inclusion $\Sigma_k \times \Sigma_\ell \to \Sigma_{k+\ell}$). The right space has as homotopy groups the stable homotopy groups of spheres. One rather highbrow way in which this is stated is:

Theorem 1.14 (Barratt-Priddy-Quillen)**.** *The K-theory[5] of finite sets is the sphere spectrum. In particular,*

$$K_i(FinSets) \cong \pi_i(\mathbb{S})$$

References

[1] Maru Sarazola. *Loop Spaces and Operads* https://drive.google.com/file/d/1_UWR66MmQWnxbYeZWB-eTQFJRbaS2eJw/view

[2] Allen Hatcher. *A Short Exposition of the Madsen-Weiss Theorem.* https://pi.math.cornell.edu/~hatcher/Papers/MW.pdf

[3] Allen Hatcher. *Algebraic Topology*.

[4] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge University Press, 1985.

[4](This analysis is nontrivial and worth its own set of notes; perhaps I will write them up at some point).

[5]The reader unacquainted with K-theory should take this as a motivating example that demonstrates the depth of the field.