The EHP Spectral Sequence

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In a first look at spectral sequences, one generally restricts to studying homology and cohomology groups of a space X with an associate filtration. However, within the general machinery, there is in nothing that particularly demands that we study the homology groups $H_n(X_i)$, rather than the homotopy groups $\pi_n(X_i)$.[1] For pairs (X_i, X_{i-1}) we have a sequence

$$\cdots \to \pi_n(X_i, X_{i-1}) \to \pi_{n-1}(X_{i-1}) \to \pi_{n-1}(X_i) \to \pi_{n-1}(X_i, Xi-1) \to \cdots$$

Furthermore, given a filtration $\bigcup X_i \cong X$, we also have that $\mathrm{colim}_i \pi_n(X_i) \cong \pi_n(X)$. Therefore, the arguments presented in Hatcher or in the accompanying genera machinery notes apply: we have a spectral sequence which starts with $\pi_n(X_{i+1}, X_i)$ and converges to the homotopy groups of X. Unfortunately, in contrast with the homological situation, relative homotopy groups are not easily computed. However, if we suppose that the map $X_i \to X_{i+1}$ is a fibration with fiber F_i, then we can connect the long exact sequence of the pair (X_{i+1}, X_i) with the long exact sequence of the fibration $F_i \to X_i \to X_{i+1}$ and derive that $\pi_n(X_{i+1}, X_i) \cong \pi_{n-1}(F_i)$. This insight will be key to constructing the EHP spectral sequence. Before doing so, we will say a little more informally and formally about the general machinery. The following section is my own attempt at unifying several phenomena, and requires some knowledge of abstract homotopy theory. It is not necessary to understand anything else in this note, and so can be skipped by a reader without this background.

1 The homological spectral sequence as a special case of the homotopical

In the first section, we constructed an exact couple whose terms were $H_n(X_i)$ and $H_n(X_i, X_{i-1})$. Recall that these relative homology groups are equivalently the homology groups of the cofiber of the inclusions $\iota_i : X_{i-1} \to X_i$. These assemble into the diagram

$$\begin{tikzcd} \cdots \arrow[r, "\iota_{i-1}"] & X_{i-1} \arrow[r, "\iota_i"] \arrow[d] & X_i \arrow[r, "\iota_{i+1}"] \arrow[d] & X_{i+1} \arrow[r, "\iota_{i+2}"] \arrow[d] & \cdots \\ & \mathrm{Cof}(\iota_{i-1}) & \mathrm{Cof}(\iota_i) & \mathrm{Cof}(\iota_{i+1}) & \end{tikzcd}$$

Now, for any space Y, one way of viewing the homology groups $H_n(Y)$ is as the homotopy groups of the space $\pi_n(\mathbb{Z}[Y])$, where $\mathbb{Z}[Y]$ denotes the free topological abelian group on Y if by "space" we mean "topological space," and denotes the free simplicial abelian group on Y if we instead mean "simplicial set." The latter is better behaved, so this is what we will work with. Now, the functor $\mathbb{Z}[-]$ preserves cofibre sequences, and therefore

$$\mathbb{Z}[X_{i-1}] \xrightarrow{\mathbb{Z}[\iota_i]} \mathbb{Z}[X_i] \longrightarrow \mathbb{Z}[\mathrm{Cof}(\iota_{i-1})]$$

is a cofibre sequence. The category of simplicial abelian groups has the extremely useful property of being *stable*, and in particular all cofibre sequences are also fibre sequences. It then follows that we have fibre sequences

[1] Other than the fact that computing homotopy groups is a generally futile endeavor.

$$\Omega\mathbb{Z}[\mathrm{Cof}(\iota_{i-1})] \longrightarrow \mathbb{Z}[X_{i-1}] \xrightarrow{\mathbb{Z}[\iota_i]} \mathbb{Z}[X_i]$$

And we can assemble these into the diagram

$$\begin{tikzcd}
 & \Omega\mathbb{Z}[\mathrm{Cof}(\iota_i)] \arrow[d] & \Omega\mathbb{Z}[\mathrm{Cof}(\iota_{i+1})] \arrow[d] & \Omega\mathbb{Z}[\mathrm{Cof}(\iota_{i+2})] \arrow[d] & \\
\cdots \arrow[r, "\mathbb{Z}[\iota_{i-1}]"] & \mathbb{Z}[X_{i-1}] \arrow[r, "\mathbb{Z}[\iota_i]"] & \mathbb{Z}[X_i] \arrow[r, "\mathbb{Z}[\iota_{i+1}]"] & \mathbb{Z}[X_{i+1}] \arrow[r, "\mathbb{Z}[\iota_{i+2}]"] & \cdots
\end{tikzcd}$$

Now, the long exact sequence of homotopy groups of a fibration yield the long exact sequences

$$\cdots \to \pi_n(\Omega\mathbb{Z}[\mathrm{Cof}(\iota_i)]) \to \pi_n(\mathbb{Z}[X_{i-1}]) \to \pi_n(\mathbb{Z}[X_i]) \to \pi_{n-1}(\Omega\mathbb{Z}[\mathrm{Cof}(\iota_i)]) \to \cdots$$

These assemble into an exact couple, which we call the **homotopical exact couple** of the filtration. From this we can repeat the same analysis as in the previous section and derive a spectral sequence. In fact, we would derive the exact same spectral sequence as before. In this context, the main theorem is this:

Theorem 1.1. *Let X be a space with a filtration $\bullet \subset X_0 \subset \cdots \subset X$, such that the homotopical exact couple satisfies (1). Then there is a spectral sequence with $E^1_{n,i} = \pi_n(Fib(\iota_i))$ and $E^\infty_{n,i} \Rightarrow \pi_n(X)$. The differentials d^r have degree $(-1, -1-r)$.*

From this point of view, the homology spectral sequence is a special case of this homotopy spectral sequence. Although we won't discuss details, there is a cohomology spectral sequence, which is dual to to the homotopy spectral sequence, not a special case (except by duality, i.e. in the same way cofibre sequences are fibre sequences in the opposite category).

2 Constructing the EHP sequence

The EHP sequence is the homotopical spectral sequence associated to the filtration

$$S^0 \to \Omega S^1 \to \Omega^2 S^2 \to \Omega^3 S^3 \to \Omega^4 S^4 \to \cdots \to \Omega^\infty S^\infty$$

Where $\Omega^\infty S^\infty$ is defined to be the colimit of the $\Omega^n S^n$. The maps of the filtration are the n-fold looping of the unit maps $S^n \to \Omega S^{n+1}$. The maps $\Omega^n S^n \to \Omega^{n+1} S^{n+1}$ are not fibrations, but this is little matter: we can take their *homotopy fibre*, denoted A_n, which will have the same property of fitting into a long exact sequence of homotopy groups as though we had a fibration $A_n \to \Omega^n S^n \to \Omega^{n+1} S^{n+1}$. Then we have a spectral sequence with $E_1^{i,n} \cong \pi_i(A_n)$, and converging to $\pi_i(\Omega^\infty S^\infty)$. Of course, all of this is little use if we cannot identify what A_n is. This is the subject of the next section.

2.1 Identifying the fibre

We wish to find A_n in the fibre sequence $A_n \to \Omega^n S^n \to \Omega^{n+1} S^{n+1}$. We will use two facts to adjust slightly our goal: firstly, that if $A \to B \to C$ is a fibre sequence, so is $\Omega C \to A \to B$. In particular, if we have that $\Omega^n S^n \to \Omega^{n+1} S^{n+1} \to C$ is a fibre sequence, so is $\Omega C \to \Omega^n S^n \to \Omega^{n+1} S^{n+1}$, so for A_n we can take ΩC. Secondly, that Ω preserves fibre sequences. Because of this, it suffices to find a space C and a map $\Omega S^{n+1} \to C$ with homotopy fibre S^n; applying these two facts will then tell us that $\Omega^{n+1} C \cong A_n$. So we seek such a space C and such a map.

To any connected topological space X with chosen basepoint e, there is a construction JX called the *James Construction* of X. To construct it, we construct a sequence

$$J_iX = \frac{X^i}{(x_1, \cdots, x_a, e, \cdots, x_i) \sim (x_1, \cdots, e, x_a, \cdots, x_i)}$$

We think of J_iX as being i-fold products of points of X, modulo the relation that the basepoint e commutes with everything. Each J_iX includes into $J_{i+1}X$ by appending an e to an i-tuple to get an $(i+1)$-tuple; if we think of concatenation as multiplication, then this expresses that e is the identity. We then let

$$JX := \operatorname{colim}\Big(J_1X \to J_2X \to J_3X \to \cdots \Big)$$

We denote points in JX by finite strings of points in X, eg $x_1x_2x_3$ denotes the point $(x_1, x_2, x_3, e, e, ...)$. JX is a topological monoid by the map which concatenates strings. Its identity is $(e, e, ...)$. JX will be useful for us because of the following fact:

Theorem 2.1. *Let X be connected. Then JX is homotopy equivalent to $\Omega\Sigma X$.*

For a proof, see [2]. The essential idea is that JX is the free associative monoid on the space X, while $\Omega\Sigma X$ is the free grouplike "associative-up-to-coherent-homotopy" monoid on the space X.[2] It turns out that the associative/associative-up-to-coherent-homotopy distinction is not relevant (see maru), and that any connected monoid is grouplike, so these coincide when X is connected.

As JX has a rather combinatorial description, this equivalence gives a surprisingly easy way to analyze a loop space. For instance, if we take X to be the sphere S^n, then we have that J_kS^n is obtained by gluing a single kn-cell to $J_{k-1}S^n$. From this we immediately obtain the following for $n \geq 2$:

$$H_i(\Omega S^{n+1}) \cong H_i(JS^n) \cong \begin{cases} \mathbb{Z} & \text{if } i|n \\ 0 & \text{otherwise} \end{cases}$$

Applying the universal coefficient theorem, we obtain the same result on cohomology. In fact, we can say somewhat more, and calculate the multiplicative structures involved here:

Theorem 2.2. *For n even, $H^*(\Omega S^{n+1}) \cong \Gamma[x]$ for x in degree n. For n odd, $H^*(\Omega S^{n+1}) \cong H^*(S^n) \otimes H^*(\Omega S^{2n+1})$.*

Proof. We inspect the cohomological Serre spectral sequence for the fibration $\Omega S^{n+1} \to PS^{n+1} \to S^{n+1}$. The E^2 page will have terms $H^p(S^{n+1}; H^q(\Omega S^{n+1}))$; in other words it will have $H^*(\Omega S^{n+1})$ in rows 0 and $n+1$.= In order for the E^∞ page to end up at 0, we must have everything get killed by differentials. In particular, some differential must hit the element a which generates $H^{n+1}(S^{n+1}, H^0(\Omega S^{n+1})) \cong \mathbb{Z}$. For degree reasons, the only possible way for this to happen is if there is an element $x_1 \in H^0(S^{2n-1}; H^n(\Omega S^{n+1}))$, and furthermore that x_1 freely generates this group. Of course, this lets us know about an additional group: $H^{n+1}(S^{n+1}; Hn(\Omega S^{n+1}))$, generated by $a \smile x_1$. So likewise, there must be some element x_2 which hits this, and then we know there is an element $a \smile x_3$... so inductively we recover what we noted above, that $H^i(\Omega S^{n+1}) \cong \mathbb{Z}$ when i divides n, and is 0 otherwise.

In addressing the multiplicative structure, we split into cases: first, suppose that n is even. Then we have that $d_n(x_1^k) = k(a \smile x_1^{k-1})$, and we already know (by definition, essentially) that $d_n(x_k) = a \smile x_{k-1}$. If we let $k = 2$, we have that $d_n(x_1^2) = 2(a \smile x_1)$, whereas $d_n(x_2) = a \smile x_1$, and so we derive $x_1^2 = 2x_2$. Inducting, we derive that in general, $x_1^k = k!x_k$, so $H^*(\Omega S^{n+1})$ is a divided power ring as we claimed. □

[2] By "grouplike" we mean that π_0 is a group; see the the author's writeup on the bar spectral sequence for more detail. By "associative-up-to-coherent-homotopy" we mean an A_∞-algebra, see [1].

Consider $J_2 S^n$. It has a subspace $J_1 S^n \cong S^n$, and the quotient $J_2 S^n / J_1 S^n$ is a sphere S^{2n}. We denote this quotient map with an overline, writing $x_1 x_2 \mapsto \overline{x_1 x_2}$. Composing this with the inclusion $S^{2i} \cong J_1 S^{2i} \hookrightarrow JS^{2i}$ yields a map $J_2 S^i \to JS^{2i}$. We extend this to a map $f : JS^i \to JS^{2i}$ by

$$f(x_1 x_2 \cdots x_n) = \prod_{1 \le i < j \le n} \overline{x_i x_j}$$

Note that this induces an isomorphism on H^{2n}, so determines the map on the homology ring. Denote by F_n the homotopy fibre of this map, so that we have a fibre sequence $F_n \to JS^n \to JS^{2n}$. Equivalently, we a have a fiber sequence $F_n \to \Omega S^{n+1} \to \Omega S^{2n+1}$. If we can show that $F_n \cong S^n$, we will have found the fiber A_n that we were originally seeking. Note that by the LES in homotopy groups, we have that $H^n(F_n) \cong \mathbb{Z}$. To understand the overall homotopy type of this F_n, we consider the cohomological Serre spectral sequence for this fibration: the E^2 page looks as follows:

$$\begin{array}{c|ccccc}
2n & ? & ? & ? & ? & ? \\
\vdots & 0 & 0 & 0 & 0 & 0 \\
n & a_0 t & a_1 t & a_2 t & a_3 t & a_4 t \\
\vdots & 0 & 0 & 0 & 0 & 0 \\
0 & a_0 & a_1 & a_2 & a_3 & a_4 \\
\hline
 & 0 & 2n & 4n & 6n & 8n
\end{array}$$

Here we represent by a_i the generator of $H^{2ni}(JS^{2n}, \mathbb{Z})$, and by t the generator of $H^n(F_n) \cong \mathbb{Z}$. By putting a group element in a spot on a spectral sequence diagram, we mean that that element generates the group there. For instance, $a_3 t$ generates $E^2_{6n,n} = H^{6n}(S^{2n}, H^n(S^n))$. In this particular groups, these elements generate these groups freely; that is, each group is isomorphic to $\mathbb{Z}$.

How is it that we know about all those groups which are 0? Well, by $(n-1)$-connectedness of the space F_n, rows 1 through $n-1$ vanish. We know that after termination of the spectral sequence, we will only have nonzero entries whose total degree is a multiple of n, because this is where the homology of JS^n is concentrated. This tells us that there can be no entries in rows $n + 1$ through $2n - 1$, since the leftmost entry in the lowest such row would survive to the end of the spectral sequence for degree reasons. Hence the first possible nonzero entry is in row $2n$.

In the case where n is odd, the elements shown represent all of the cohomology of JS^n (the inclusions of these groups are isomorphisms) and consequently they support no differentials. Since these support no differentials, we have that there are actually no nonzero terms above these (as any such terms would survive to the E^∞ page), and hence F_n is a simply connected cohomology sphere, and hence homotopy equivalent to a sphere, the homotopy equivalence being induced by the generator of $\pi_n(F_n) \cong \mathbb{Z}$

In the case that n is even, however, these rows do not represent all of the cohomology of JS^n: in particular, the element $a_1 t$ is sent to the product of the generators in dimension n and $2n$ of $H^*(JS^n)$. Due to the divided power structure, this is three times a generator of this group, to reflect this filtration $\langle a_t \rangle \subset H^{3n}(JS^n)$ which has subquotient $\mathbb{Z}/3\mathbb{Z}$, the first entry in row $3n$ is nonzero, equal to $\mathbb{Z}/3\mathbb{Z}$ (all entries in rows $n + 1$ through $3n - 1$ are zero). However, if we rerun the spectral sequence argument using rational homology rather than singular homology, then there is no such problem and we obtain the result that F_n is a rational cohomology sphere. Additionally, if we work with coefficients in $\mathbb{Z}/2\mathbb{Z}$, this divided power disappears and we similarly get that A_n is a $\mathbb{Z}/2\mathbb{Z}$-cohomology sphere. From this we get that F_n is a 2-local cohomology sphere, and hence 2-locally homotopy equivalent to a sphere (here we must also appeal to the fact that the homotopy groups of F_n are finitely generated, which follows from finite generation of the homotopy groups of spheres and that it appears in a fibration with them).[3]

At this point, our analysis has yield that we have a 2-local fibre sequence $S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$. We can take the n-fold looping of this to get a fibre sequence $\Omega^n S^n \to \Omega^{n+1} S^{n+1} \to \Omega^{n+1} S^{2n+1}$. This tells us that we have a fibre sequence $\Omega^{n+2} S^{2n+1} \to \Omega^n S^n \to \Omega^{n+1} S^{n+1}$. The map $\Omega^n S^n \to \Omega^{n+1} S^{n+1}$ is the one occuring in the filtration of $\Omega^\infty S^\infty$ (this can be determined by analyzing the effect on cohomology). Hence the general method of constructing a spectral sequence from a filtration gives us the following:

Theorem 2.3. *There is a 2-local spectral sequence with E^1 page given by $E^1_{n,i} \cong \pi_i(\Omega^{n+2} S^{2n+1}) \cong \pi_{i+n+2}(S^{2n+1})$ converging to $\pi_i(\Omega^\infty S^\infty)$.*

We haven't solved the overall problem of finding the homotopy groups of F_n, but finding them 2-locally is a rich enough topic on its own that we will restrict our attention to it for the rest of this section.

CONVENTION ALERT: THE REMAINDER OF THIS DOCUMENT IS 2-LOCAL; THAT IS, ALL STATEMENTS SHOULD BE INTERPRETED AS BEING TRUE ONLY AFTER LOCALIZATION AT 2

3 Understanding the spectral sequence

The spectral sequence that we have just described is known as the EHP Sequence. At first glance, it's a bit of a silly construction: the input is the biggest open problem in homotopy theory, and the output is the second biggest. So we can neither reason backwards nor forwards to "solve" this spectral sequence. However, it is a very useful organizational method for dealing with some fascinating structure in the homotopy groups of

[3] For the reader unfamiliar with seeing "local" in the context of topological spaces: acquaint yourself with the word in the context of abelian groups (or better yet, modules over a general commutative ring). A "2-local sphere" is a space X equipped with a map $S^k \to X$ which is an equivalence on all the 2-localized homotopy groups).

spheres. To explain this, we first explain the name: the "E" stands for *Einhangung*, the German word for "suspension," and refers to the map $\Omega^n S^n \to \Omega^{n+1} S^{n+1}$. The "H" stands for Hopf, and is motivated by the following observation:

Theorem 3.1. *The map $\pi_{2n-1}(S^n) \to \pi_{2n-1}(S^{2n-1})$ is the Hopf invariant map, possibly up to a negative sign.*

The proof is rather long and computational, so we simply reference [2]. Well, that's the "H," and we'll explore some implications in a bit. For now, let's move onto the "P:" this stands for Product, and it is in reference to the Whitehead Product. For completeness, we include a definition:

Definition For $\alpha \in \pi_i(X)$ and $\beta \in \pi_j(X)$, the Whitehead product $[\alpha, \beta]$ is defined as follows: the space $S^i \times S^j$ is the pushout of the following diagram:

$$\begin{tikzcd} S^{i+j-1} \arrow[r, hook, "\iota"] \arrow[d, "A"] & D^{i+j} \\ S^i \vee S^j & \end{tikzcd}$$

And so we can define $[\alpha, \beta]$ to be the composition $(\alpha \vee \beta) \circ A$.

Example As we calculated before, $H^*(JS^n)$ is a divided power algebra for n even. Hence $J_2 S^n \cong Z[x, x^2/2]/(x^3)$, so the Hopf invariant of the attaching map of the $2n$-cell of $J_2 S^n$ is ± 2. This map is precisely $[\iota_n, \iota_n]$. On the other hand, if n is odd, the generator in dimension n squares to 0 and so we obtain

$$H([\iota_n, \iota_n]) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \pm 2 & \text{for } n \text{ even} \end{cases}$$

This is the classical definition of the Whitehead product, we state another formulation which we will not use but feel is important to mention:

Theorem 3.2. *Let X have the homotopy type of a CW complex. Consider the commutator map $[-,-] : \Omega X \times \Omega X \to \Omega X$ given by $(p, q) \mapsto pqp^{-1}q^{-1}$. The image of $\Omega X \vee \Omega X$ under this map is contractible, and hence we have an induced map $\Omega X \wedge \Omega X \to \Omega X$. Consider the composition*

$$S^{i+j} \longrightarrow S^i \wedge S^j \xrightarrow{\alpha \wedge \beta} \Omega X \wedge \Omega X \xrightarrow{asdf} \Omega X$$

This yields a map $[-,-] : \pi_i(\Omega X) \times \pi_j(\Omega X) \to \pi_{i+j}(\Omega X)$, which is equivalent to the Whitehead bracket under the identification $\pi_n(\Omega X) \cong \pi_{n+1}(X)$. Furthermore, this makes ΩX into a graded Lie algebra over $\mathbb{Z}$.

The Lie algebra structure is the reason for the bracket notation, and because of this the product is also referred to as the Whitehead *bracket*. But what does this have to do with our spectral sequence? Well, if we write out the following exact fragment:

$$\pi_{2n+1}(S^{2n+1}) \to \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1}) \to 0$$

We claim the leftmost map has something to do with the Whitehead product. The middle map is an "E." By the equivalence between $\Omega\Sigma X$ and JX that we discussed earlier, this is isomorphic to the diagram

$$\pi_{2n+1}(S^{2n+1}) \to \pi_{2n-1}(S^n) \to \pi_{2n-1}(JS^n) \to 0$$

where the middle map is induced by the inclusion $S^n \to JS^n$. By homotopy excision, the kernel of this map is generated by the attaching map of the $2n$ cell of JS^n. This is exactly the Whitehead product $[\iota_n, \iota_n]$. By exactness, the leftmost map sends a generator of $\pi_{2n+1}(S^{2n+1})$ to $[\iota_n, \iota_n]$, and this is the connection we were seeking.

This last observation has particular significance for calculating the differentials along the diagonal: these differentials (on the E^1 page) are the composition $H \circ P$, and so along the diagonal are given by the Hopf invariant of $[\iota_n, \iota_n]$. In particular, as we noted in the example above, they alternate between 2 and 0. Furthermore, reviewing the definition of the spectral sequence of an exact couple, we obtain the following:

Theorem 3.3. *The differential d^i with source $\pi_{2n-1}(S^{2n-1})$ is 0 if $i < \rho(n)$, where $\rho(n)$ is the number of times one can "desuspend" $[\iota_n, \iota_n]$.*

This number $\rho(n)$ turns out to have some surprising geometric connections:

Theorem 3.4. *The maximal number of linearly independent vector fields on S^n (equivalently, the dimension of the largest trivial summand of the tangent bundle of S^n) is equal to $\rho(n)$.*

We defer this proof to a short appendix. As a consequence, the term $\pi_7(S^7)$ supports no differentials, just for dimension reasons (since S^7 is parallelizable).

3.1 Calculating some stable (and unstable) homotopy groups

Now, we will try to gather together these facts and compute some stable homotopy groups: let us start with the baseline knowledge that $\pi_{2n-1}(S^{2n-1}) \cong \mathbb{Z}$. This tells us that on the E^1 page, we have

$$\begin{tikzcd}
S^7 & 0 & 0 & 0 & \mathbb{Z} \arrow[dl] & ? \\
S^5 & 0 & 0 & \mathbb{Z} \arrow[dl] & ? & ? \\
S^3 & 0 & \mathbb{Z} \arrow[dl] & ? & ? & ? \\
S^1 & \mathbb{Z} & 0 & 0 & 0 & 0 \\
 & \pi_0^s & \pi_1^s & \pi_2^s & \pi_3^s & \pi_4^s
\end{tikzcd}$$

The differentials drawn alternate between 0 and 2, and so in particular we calculate that $\pi_0^s \cong \mathbb{Z}$ and $\pi_1^s \cong \mathbb{Z}/2\mathbb{Z}$.[4] But $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ in the stable range, so we get to fill in some of the E^1 page to get:

$$
\begin{tikzcd}
S^7 & 0 & 0 & 0 & \mathbb{Z} \arrow[dl] & \mathbb{Z}/2\mathbb{Z} \\
S^5 & 0 & 0 & \mathbb{Z} \arrow[dl] & \mathbb{Z}/2\mathbb{Z} & ? \\
S^3 & 0 & \mathbb{Z} \arrow[dl] & \mathbb{Z}/2\mathbb{Z} & ? & ? \\
S^1 & \mathbb{Z} & 0 & 0 & 0 & 0 \\
 & \pi_0^s & \pi_1^s & \pi_2^s & \pi_3^s & \pi_4^s
\end{tikzcd}
$$

And now we can attempt to analyze the behaviour of the differentials. Denote by η_3 the generator of $\pi_4(S^3)$; we would like to prove that no differentials kill it. The differential on the E^2 page, coming from ι_7, cannot hit it because S^7 is parallelizable. To analyze the other differential, we make use of the following identity:

Lemma 3.5. *Let α and β be elements in the homotopy groups of spheres. Then*

$$P(\alpha \circ E^2\beta) = P(\alpha) \circ \beta$$

when these compositions are defined.

This is useful because if we let $\alpha = \iota_5$ and $\beta = \eta_3 : S^4 \to S^3$, then we get that $P(\eta_5) = [\iota_2, \iota_2] \circ \eta_3$, but η_3 is order 2 and $[\iota_2, \iota_2]$ is twice η_2, so this is 0, hence the term η_5 supports no differential. As a consequence, we calculate that $\pi_2^s \cong \mathbb{Z}/2\mathbb{Z}$. Actually, we can observe by inspecting our LES that here we have early stabilization in the homotopy groups of spheres: we have the following exact fragment

$$\pi_7(S^4) \xrightarrow{H} \pi_7(S^7) \xrightarrow{P} \pi_5(S^3) \xrightarrow{E} \pi_6(S^3) \xrightarrow{H} \pi_6(S^7)$$

The last H is 0 for degree reasons, and the first H is surjective because of the Hopf invariant one map $\nu : S^7 \to S^4$. It follows that $\pi_5(S^3) \cong \pi_6(S^4) \cong \pi_2^s$. and we get in particular $\pi_5(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. Note that similar logic tells us that in general, the only time early stabilization can occur in the homotopy groups of spheres is when H is surjective, i.e. when there is a map of Hopf invariant one.

[4] Every other calculation is only valid 2-locally, but actually this holds integrally: for, the Hopf invariant of $[\iota_2, \iota_2]$ is 2, and hence $2\eta = 0$ stably where η is the Hopf fibration.

At this point, we've deduced that the E^1 page looks as follows:

$$
\begin{tikzcd}
S^7 & 0 & 0 & 0 & \mathbb{Z} \arrow[dl] & \mathbb{Z}/2\mathbb{Z} \\
S^5 & 0 & 0 & \mathbb{Z} \arrow[dl] & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
S^3 & 0 & \mathbb{Z} \arrow[dl] & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & ? \\
S^1 & \mathbb{Z} & 0 & 0 & 0 & 0 \\
 & \pi_0^s & \pi_1^s & \pi_2^s & \pi_3^s & \pi_4^s
\end{tikzcd}
$$

And that no non-diagonal elements support a differential in first 4 columns. To calculate π_3^s, we will show that no elements in the fifth column support a differential, except for the differential $\pi_9(S^9) \to \pi_7(S^7)$ (which is just outside of the portion of the page which we've included). We consider each of these elements in turn (note that for degree reasons, we do not need to consider any differentials from the question mark at the bottom right).,

The $\mathbb{Z}$ on the diagonal vanishes after the E_1 page, since the d^1 differential coming out of it is multiplication by two.

Next, the $\mathbb{Z}/2\mathbb{Z}$ generated by $\Sigma^5\eta$: for degree reasons, it suffices to show that d^1 and d^2 are 0, so we wish to show that $P(\Sigma^6\eta)$ is a 2-fold suspension. Using the previous formula,

$$
\begin{aligned}
P(\Sigma^5\eta) &= P(\iota^7 \circ \Sigma^5\eta) \\
&= P(\iota_7) \circ \Sigma^3\eta
\end{aligned}
$$

Note that since ι_7 supports no differentials, $P(\iota_7)[= [\iota_3, \iota_3]$ is a 3-fold suspension. Since $\Sigma(\alpha \circ \beta) = \Sigma\alpha \circ \Sigma\beta$, the result then follows.

Finally, the $\mathbb{Z}/2\mathbb{Z}$ generated by $\Sigma^3\eta^2$: for degree reasons, we need only show that d^1 vanishes. Using again using our formula, we have

$$\begin{aligned}(H \circ P)(\Sigma^3\eta^2) &= H(P(\Sigma^3\eta \circ \Sigma^4\eta)) \\ &= H(P(\Sigma^3\eta) \circ \Sigma^2\eta) \\ &= H(P(\Sigma^3\eta)) \circ \Sigma^2\eta \\ &= d^1(\Sigma^3\eta) \circ \Sigma^2\eta \\ &= 0 \circ \Sigma^2\eta = 0\end{aligned}$$

So we find that all three $\mathbb{Z}/2\mathbb{Z}$ terms in the π_3^s column survive, so we have a filtration

$$0 \cong \pi_4(S^1) \to \pi_5(S^2) \to \pi_6(S^3) \to \pi_7(S^4) \to \pi_8(S^5) \to \pi_3^s$$

Where the subquotients of the images of the terms in the final term π_3^s are $\mathbb{Z}/2\mathbb{Z}$. This means that π_3^s is either $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^3$. Reconstructing the fact that it is in fact $\mathbb{Z}/8\mathbb{Z}$ requires a surprising amount of additional work that we will not do here.

4 Further Directions

While our focus has been on constructing and analyzing this particular spectral sequence, it is worth saying some words on closely related topics. We present no proofs for the following observations; some are uncomplicated but some are deep and difficult. Most of them are known to the author through [4]. First of all, if we are interested not in computing the stable homotopy groups of spheres, but rather the homotopy groups of S^n, we can consider the filtration

$$S^0 \to \Omega^1 S^1 \to \cdots \to \Omega^n S^n \to \Omega^n S^n \to \cdots$$

and we will obtain a spectral sequence which is trivial above the nth row, converging to the homotopy groups of S^n. This is a sort of "truncated EHP Sequence."

There is a glaring deficiency of our analysis: it is entirely blind to odd primes! Fortunately, there is similar analysis that we can do - for an odd prime p, Toda constructed some more intricate p-local fibre sequences

$$\hat{S}^{2m} \to JS^{2m} \to JS^{2mp} \qquad \text{and} \qquad S^{2m-1} \to \Omega\hat{S}^{2m} \to JS^{2pm-2}$$

where p is an odd prime. Here $\hat{S}^{2m}$ denotes the $(2mp-1)$-skeleton of JS^{2m}. Gluing these together yield arrays of spectral sequences which converge to $\pi_\bullet(S^n)$ for n odd, and $\pi_\bullet(\hat{S}^n)$ for n even.

Another interesting related tidbit is the following: by viewing S^n as the one point compactifiation of $\mathbb{R}^n$, we have an action of $O(n)$ on S^n - in other words, a map $O(n) \to \Omega^n S^n$. This is compatible with the inclusion $O(n) \to O(n+1)$. Furthermore, this inclusion is the fiber of a map $O(n+1) \to S^{n+1}$. Fitting this together, there is a commutative diagram

$$\begin{tikzcd} O(n) \arrow[r] \arrow[d] & O(n+1) \arrow[r] \arrow[d] & S^n \arrow[d] \\ \Omega^n S^n \arrow[r] & \Omega^{n+1} S^{n+1} \arrow[r] & \Omega^{n+1} S^{2n+1} \end{tikzcd}$$

Where the rows fibre sequences. The top row has its own spectral sequence, converging to the homotopy groups of O, the colimit of the $O(n)$. The vertical maps induce a map of spectral sequences, which is compatible with the J homorphism $O \to \Omega^\infty S^\infty$. This is related to our two equivalent descriptions of $\rho(n)$ (the construction of the fibre sequence is a key element of our proof of their equivalence). It turns out that we can further enlarge this to include another row

$$\begin{tikzcd}
O(n) \arrow[r] \arrow[d] & O(n+1) \arrow[r] \arrow[d] & S^n \arrow[d] \\
\Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[d] \\
\Omega^\infty \Sigma^\infty \mathbb{R}P^{n-1} \arrow[r] & \Omega^\infty \Sigma^\infty \mathbb{R}P^n \arrow[r] & \Omega^\infty \Sigma^\infty S^n
\end{tikzcd}$$

And this bottom row, which gives rise to a spectral sequence converging to the stable homotopy groups of $\mathbb{R}P^\infty$, also turns out to correspond to another description of ρ:

Theorem 4.1. $\rho(n)$ *is the largest integer such that the stabilization of the map* $\mathbb{R}P^n/\mathbb{R}\mathbb{P}^{n-\rho(n)} \to S^n$ *(defined by collapsing all but the top-dimensional cell) admits a cross section.*

5 Appendix: Vector Fields on Spheres

We prove here the statement that the maximal dimension of a trivial subbundle of S^n is equal to the number of times that $[\iota_n, \iota_n]$ can be desuspended.

Recall that to give an n-dimensional vector bundle on a space X, it suffices to find an open cover $\mathcal{U}$ of X such that each $U \in \mathcal{U}$ is contractible, and provide functions $\tau_{UV} : U \cap V \to \mathrm{GL}(n)$ such that $\tau_{VW} \circ \tau_{UV} = \tau_{UW}$. Furthermore, the vector bundle depends (up to isomorphism) only on the homotopy classes of the τ_{UV}. In particular, to give a vector bundle on the n-sphere, we consider the open cover by $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$, where N and S are the North and South poles. The intersection of these two open sets is homotopy equivalent to a S^{n-1}; hence giving a vector bundle on S^n is equivalent to giving an element $\pi_{n-1}(\mathrm{GL}_n)$. Denote the element corresponding to TS^n by ν_n. We can (somewhat) explicitly provide a description of ν_n:

Lemma 5.1. $J(\nu_n) = \pm[\iota_n, \iota_n]$.

I couldn't figure out a better proof of this than the original, so the reader interested in the proof should see it at [3]. With this in hand, we can prove our desired result:

Proof of Theorem 3.7. By the above, there exists an n-dimensional trivial subbundle of TS^n iff ν_n is in the image of the inclusion $\pi_{n-1}(SO(n-r)) \to \pi_{n-1}(SO(n))$. Looking at the commutative diagram:

$$\begin{array}{ccc} \pi_{n-1}(SO(n-r)) & \longrightarrow & \pi_{n-1}(SO(n)) \\ \downarrow & & \downarrow \\ \pi_{n-1}\Omega^{n-r}S^{n-r} & \longrightarrow & \pi_{n-1}\Omega^n S^n \end{array}$$

We see that if ν_n is in the image of the top horizontal map, then $[\iota_n, \iota_n]$ is in the image of the bottom horizontal map. For the other direction: there is a generalized Hopf invariant $\pi_{n-1}\Omega^n S^n \to \pi_{n-1}(SO(n), SO(n-r))$ which is 0 exactly on r-fold suspensions. Hence if $[\iota_n, \iota_n]$ is an r-fold suspension, ν_n is sent to zero in the composition

$$\pi_{n-1}SO(n) \to \pi_{n-1}\Omega^n S^n \to \pi_{n-1}(SO(n), SO(n-r))$$

This composition is equal to the map occuring in the LES of the pair $(SO(n), SO(n-r))$ and therefore this implies ν_n is in the image of $\pi_{n-1}SO(n-r)$ as desired. $\square$

References

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