Introduction

Have this equivalence for compact, Hausdorff spaces X:

vector bundle over X $\longrightarrow$ Projective module over C(X)

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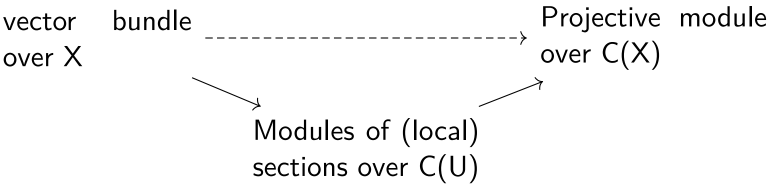

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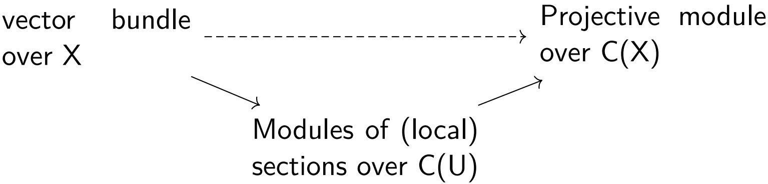

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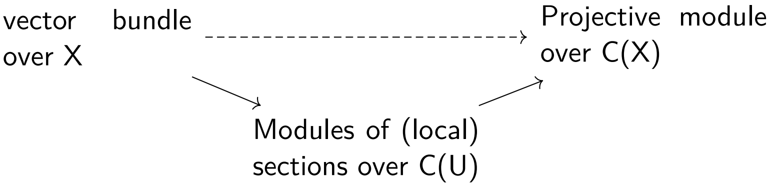

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↘ Modules of (local) sections over C(U) ↗

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Bundles and Local Sections

Given a vector bundle $p : E \to X$, we can take local sections on any open subset $U \subset X$. These form a module over the ring of functions $C(U)$.

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Question: given some suitable data of "modules over (some) of the rings C(U)," can we construct a corresponding vector bundle?

Starting point: given a finitely generated, free module over $C(X)$, it is the global sections of a trivial bundle.

What's a Vector Bundle?

A vector bundle over a space X is an open cover $\{U_i\}$ and transition homeomorphisms $f_{\alpha\beta} : U_\alpha \cap U_\beta \times \mathbb{R}^n \to U_\alpha \cap U_\beta \times \mathbb{R}^n$ (which are over X, linear on fibers, and satisfy a cocyle condition over $U_\alpha \cap U_\beta \cap U_\gamma$).

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$$
\begin{array}{ccccc}
U_\alpha \times \mathbb{R}^n & & & & U_\beta \times \mathbb{R}^n \\
& \nwarrow^{\iota} & & {}^{\iota}\nearrow & \\
& U_\alpha \cap U_\beta \times \mathbb{R}^n & \xrightarrow{f_{\alpha\beta}} & U_\alpha \cap U_\beta \times \mathbb{R}^n &
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$$

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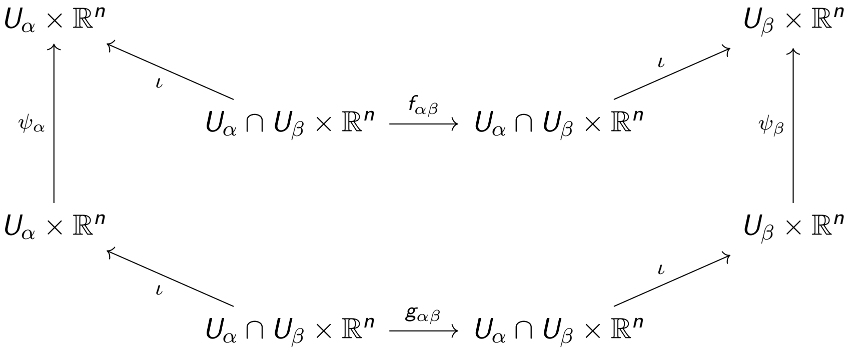

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$$C(U_\alpha)^{\oplus n} \xrightarrow{r} C(U_\alpha \cap U_\beta)^{\oplus n} \xrightarrow{\tilde{f}_{\alpha\beta}} C(U_\alpha \cap U_\beta)^{\oplus n} \xleftarrow{r} C(U_\beta)^{\oplus n}$$

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The cocycle condition $\tilde{f}_{\beta\gamma}\tilde{f}_{\alpha\beta} = \tilde{f}_{\alpha\gamma}$ makes sense viewing them all as elements of $GL_n(C(U_\alpha \cap U_\beta \cap U_\gamma))$.

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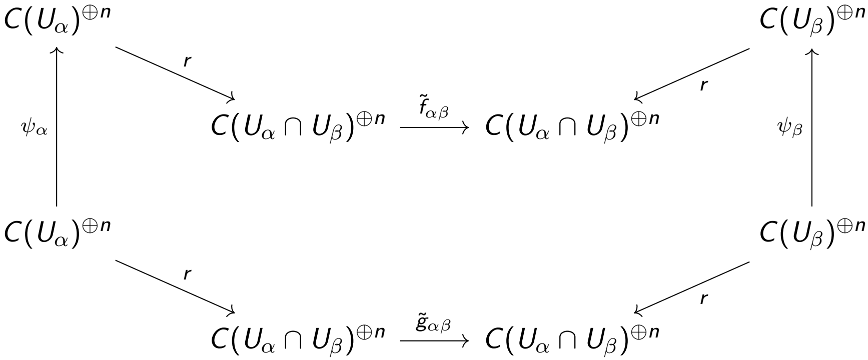
Building on our previous idea of what a map between (geometric) vector bundles is, a map between 'algebraically presented' vector bundles is a big commutative diagram something like:



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Note: the maps marked r are not surjective, but their image is a generating set (for the same reason $C(U_\alpha) \to C(U_\alpha \cap U_\beta)$ is not surjective, but its image contains 1).

Example

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The trivial line bundle is obtained by functions which are everywhere positive or everywhere negative, and the Mobius bundle by picking a function which has opposite sides on the two connected components.

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- Replace $C(U)$ with $\mathcal{O}^H(U)$ (holomorphic functions) - get holomorphic vector bundles.
- Replace $C(U)$ with $\mathcal{O}^A(U)$ (algebraic functions) - get "algebraic vector bundles."

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(Unimportant aside: this map is injective, and a group homomorphism).

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So the theorem doesn't hold in general once we use some other notion of 'function.'

Theorem (Serre's Theorem, 1955)

Let $V \subset \mathbb{C}^n$ be the solution set of the polynomials $p_1(x_1, ..., x_n) = \cdots = p_k(x_1, ..., x_n) = 0$. Algebraic vector bundles on V are equivalent to projective modules over algebraic functions on V, which is the ring $\mathbb{C}[x_1, ..., x_n]/(p_1, ..., p_k)$

K-Theory

Suggests that we should look for a homology theory for rings: an *algebraic* K-Theory on CommRing, for which the 0th functor is given by

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This is functorial because given a map $R \to S$ and a projective R-module M, the S-module $S \otimes_R M$ is projective. Problem: not so obvious how to create a homology theory for rings (can't just use spectrum!).

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No "higher Serre-Swan" theorem. Higher Algebraic K-Theory is not even (in general) periodic.

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