A couple must-know cohomology calculations

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We will follow roughly the path used in "Algebraic Topology" by Tammo tom Dieck (my personal favorite Atop reference) in order to calculate the cohomology rings of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$. The calculation there takes place in sections 17.8-17.9; the primary ingredient is the Thom Isomorphism Theorem. Hatcher was also referenced (mainly to go from $\mathbb{Z}/2\mathbb{Z}$ coefficients to $\mathbb{Z}$ coefficients for $\mathbb{RP}^n$), as well as online notes by Will Merry (mainly for the orientation bit).

By H^* we mean singular cohomology with coefficients in a ring R. All modules will be left modules. Results are proved over compact/finite dimensional CW complexes to save me some effort.

1 The Leray-Hirsch Theorem

Definition 1.1. *A **relative fibration** of the pair (E, E') over B is a map $p : E \to B$ such that both p and $p|_{E'}$ are fibrations. We will write p' for $p|_{E'}$, and write F and F' for the fibers over the base point $* \in B$ of p and p', respectively (and i and i' for their respective inclusions). We do not require that E' be nonempty.*

Consider a relative fibration $(F, F') \xhookrightarrow{(i,i')} (E, E') \xrightarrow{(p,p')} B$. Given an element $c \in H^*(E, E')$ and an element $b \in H^*(B)$, we can take the cup product $p^*(b) \smile c$. This endows $H^*(E, E')$ with the structure of an $H^*(B)$ module. We can add the structure of an $H^*(B)$ module to $H^*(F, F')$ by tensoring to form $H^*(B) \otimes H^*(F, F')$. While we in general have no map between the two, if the map i^* is surjective and $H^n(F, F')$ is free for each n, then we can define a map in the following manner: let $\{c_j\}$ be a set of elements of $H^*(E, E')$ such that $\{i^*(c_j)\}$ is a basis for $H^*(F, F')$ (as an R-module). Then the partially defined map

$$L : H^*(B) \otimes H^*(F, F') \to H^*(E, E')$$

given by $1 \otimes i^*(c_j) \mapsto c_j$ extends uniquely to a map of $H^*(B)$ modules.

Theorem 1.2. *Let $(F, F') \xhookrightarrow{(i,i')} (E, E') \xrightarrow{(p,p')} B$ be a relative fibration over a connected, finite dimensional CW complex. Suppose that there are classes $c_j \in H^*(E, E')$ such that $\{i_x^*(c_j)\}$ freely generate $H^*(F_x, F_x')$ for each $x \in B$ (as an R module). Then the map $L : H^*(B) \otimes H^*(F, F') \to H^*(E, E')$ described above is an isomorphism of graded R-modules.*

Note as a consequence that the elements $\{c_j\}$ are a basis for $H^*(E, E')$ as an $H^*(B)$ module (so it is free).

Proof. Note that for a subspace $A \subset B$, we obtain a restricted relative fibration

$$(F, F') \xhookrightarrow{(i,i')} (E|A, E'|A) \xrightarrow{(p,p')} A$$

and an associated module map $L_A : H^*(A) \otimes H^*(F, F') \to H^*(E|A, E'|A)$, where we use the restrictions of c_j to $H^*(E|A, E'|A)$. In the case $A = B^0 = \{*\}$, this reduces to

$$L_{B^0} : R \otimes H^*(F, F') \to H^*(F, F')$$

Which is just the isomorphism $1 \otimes c \mapsto 1 \smile c = c$. More generally, for $B^0 = \coprod_I e_i^0$, the map is

$$L_{B^0} : (\prod_I R) \otimes H^*(F, F') \to \prod_I H^*(E|e_i^0, E'|e_i^0)$$

Which because $H^*(F, F')$ is finitely generated and free, reduces to the isomorphism

$$\coprod_i L_{e_i^0} : \coprod_I (R \otimes H^*(F, F')) \to \coprod_I H^*(E|e_i^0, E'|e_i^0)$$

We will now proceed by induction on the skeleta B^n of B, with $n = 0$ as our base case: assume that the isomorphism holds for the fibration restricted to B^{n-1}. Let U be the subspace of B^n obtained by deleting a single point from each n-cell of B^n - then U is homotopy equivalent to B^{n-1}. Let V be the subspace of B^n consisting of the interiors of all the n-cells. Then we obtain MV sequences, which give us a big commutative diagram:

$$\begin{array}{ccccc}
H^*(U \cup V) \otimes H^*(F, F') & \longrightarrow & H^*(U) \otimes H^*(F, F') \oplus H^*(V)) \otimes H^*(F, F') & \longrightarrow & H^*(U \cap V) \otimes H^*(F, F') \\
\downarrow{\scriptstyle L_{U \cup V}} & & \downarrow{\scriptstyle L_U \oplus L_V} & & \downarrow{\scriptstyle L_{U \cap V}} \\
H^*(p^{-1}(U \cup V)) & \longrightarrow & (H^*(p^{-1}(U)) \oplus H^*(p^{-1}(V))) & \longrightarrow & H^*(p^{-1}(U \cap V))
\end{array}$$

The bottom row is exact because it is a MV sequence, and the top row is exact because $H^*(F, F')$ is free, so tensoring with it preserves exactness. Once we show that L_U, L_V, and $L_{U \cap V}$ are isomorphisms, we can invoke the five lemma to conclude that $L_{U \cup V} = L_{B^n}$ is an isomorphism, which will complete our inductive step.

For L_U: because U retracts to B^{n-1}, this follows from the induction hypothesis.

For L_V: Because $V = \coprod_I e_i^n$ and the e_i^n are contractible, this follows in the same way as the base case (although now we have that $(F, F') \to (E|e_i^n, E'|e_i^n)$ is a weak homotopy equivalence rather than simply being equal).

For $L_{U \cap V}$: $U \cap V$ deformation retracts to $\coprod_I \partial e_i^n$, a disjoint union of $(n-1)$-spheres. By the induction hypothesis, the isomorphism holds over any single cell boundary. Using the same techniques as in the L_V case, it holds over their disjoint union. □

2 Thom Classes and the Gysin Sequence

As a special case, consider a vector bundle of dimension n over a finite dimensional CW complex, $p : E \to B$. Let $E^0 = E \setminus B$ (considering B as the 0 section of E). Then $(E, E^0) \to B$ is a relative fibration with fiber $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. In this case, the cohomology of the fiber is relatively simple, being R in degrees 0 and n. Consequently, given only a single cohomology class $t \in H^n(E, E^0)$ such that its restriction to any fiber is a generator, we obtain as a consequence of the Leray-Hirsch Theorem that the map $H^k(B) \to H^{n+k}(E, E^0)$ given by $b \mapsto p^*(b) \smile t$ is an isomorphism for $0 \leq k$. Even more, p is a homotopy equivalence. This allows us to take the long exact sequence for the pair (E, E^0) and replace $H^k(E)$ with $H^k(B)$, and $H^k(E, E^0)$ with $H^{k-n}(B)$. This gives us the *Gysin sequence* for the bundle $p : E \to B$:

$$\cdots \longrightarrow H^{k-1}(E^0) \longrightarrow H^{k-n}(B) \longrightarrow H^k(B) \longrightarrow H^k(E^0) \longrightarrow \cdots$$

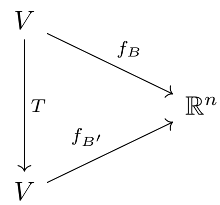
From the definitions, we get that the map $H^{k-n}(B) \to H^k(B)$ is given by $b \mapsto b \smile (p^*)^{-1}(t)$. The element $(p^*)^{-1}(t)$ we denote by e and call the *Euler class*. The element t is called a *Thom class*.

At this point, the main ingredient we are missing is an understanding of when a Thom class exists for a cohomology theory. For the rest of this section, we will focus mainly on singular cohomology with $\mathbb{Z}$ coefficients, and so H^* will mean singular cohomology with coefficients in $\mathbb{Z}$.

An orientation for a real vector space V is an equivalence class of ordered bases; two ordered bases $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ are equivalent iff the map $v_i \mapsto w_i$ has positive determinant. A *cohomological orientation* of the vector space V we will define as a generator of $H^n(V, V \setminus \{0\}) \cong \mathbb{Z}$. Note that an ordinary orientation determines a cohomological orientation in the following way: fix a generator $\gamma_n \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Then a basis $B = \{v_1, ..., v_n\}$ of V determines a homeomorphism $f_B : V \to \mathbb{R}^n$ where $v_i \mapsto e_i$. Then $f_B^*(\gamma_n)$ is a generator for $H^n(V, V \setminus \{0\})$. We denote it γ_B.

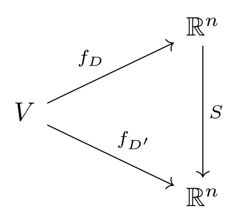
Lemma 2.1. *Two bases B and B' are the same orientation if and only if $\gamma_B = \gamma'_B$*

Proof. Firstly, suppose that B and B' are the same orientation. Then the change of basis map $T : V \to V$ taking B to B' has positive determinant, and is therefore homotopic to the identity map (since $GL_n(V)$ has two connected components). We have a commutative diagram



Since T^* is the identity map, we have that $\gamma_B = \gamma_{B'}$.

Conversely, suppose that B and B' are opposite orientations. We want to show that $\gamma_B \neq \gamma_{B'}$; by the last paragraph it suffices to show that there exist two bases of opposite orientation D and D' such that $\gamma_D \neq \gamma_{D'}$. Up till now we have not required γ_n to be represented by anything in particular; now let it be represented as the dual of the n-simplex such that the ith vertex q_i is equal to the ith standard basis element e_i for $i > 0$, and $q_0 = -\frac{1}{n}\sum e_i$. Let D be any ordered basis and D' the basis obtained by swapping the first two vectors of D. Let S be the change of basis map on $\mathbb{R}^n$ which swaps e_1 and e_2. Then the following commutes:



Since $S^* : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\} \to H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is negation, we get that $\gamma_D = -\gamma_{D'}$ ☐

To carry this over to the case of vector bundles, consider a trivial vector bundle $U \times \mathbb{R}^n \to U$. The projection $\pi : U \times \mathbb{R}^n \to \mathbb{R}^n$ yields a class $\pi^*(\gamma_n) \in H^n(U \times \mathbb{R}^n, U \times \{0\})$. For any point $x \in U$, we have an inclusion $\iota_x : \{x\} \times \mathbb{R}^n \hookrightarrow U \times \mathbb{R}^n$ and see that $\iota_x^*(\pi^*(\gamma_n))$ is a generator. Therefore, $\pi^*(\gamma_n)$ is a Thom class for this bundle.

Now suppose we have a vector bundle $p : E \to B$ and trivializations φ_U and φ_V over U and V, with $U \cap V \neq \emptyset$. Then for any point $x \in U \cap V$ we have generators $\iota_x^*(\pi_V^*(\gamma_n))$ and $\iota_x^*(\pi_U^*(\gamma_n))$. We will call the trivializations U and V *compatible* if these two generators are equal for all such x.

Lemma 2.2. *A vector bundle $p : E \to B$ over a compact CW complex has a Thom class for $\mathbb{Z}$ coefficients iff there is a trivialization such that all pairs of intersecting sets are compatible.*

Proof. Suppose E has a Thom class, denoted t. Fix some trivializing cover of E such that all the open sets are connected. Let $U \subset B$ be an element of our trivialization. Then $\iota_U^*(t)$ is a generator of $H^n(p^{-1}(U), p^{-1}(U) \setminus U) \cong \mathbb{Z}$ (by the relative suspension isomorphism). Note that $\pi_U^*(\gamma_n)$ is also a generator; we change the trivialization φ_U if necessary (by a sign in one coordinate, say) to ensure that $\pi_U^*(\gamma_n) \cong \iota_U^*(t)$. This will ensure compatibility: let $V \subset B$ be another element of the trivialization and $x \in U \cap V$; we have that $\iota_x(\pi_U^*(\gamma_n)) = \iota_x(\pi_V^*(\gamma_n)) = \iota_x^*(t)$.

Conversely, suppose that we have trivializing cover such that all intersecting pairs are compatible. We will use a Mayer-Vietoris sequence to paste together the classes of the form $\pi_U^*(\gamma_n)$ into a Thom class: given U and V in the trivializing cover, we get an exact sequence

$$H^{n-1}(p^{-1}(U \cap V), U \cap V) \longrightarrow H^n(p^{-1}(U \cup V), U \cup V) \longrightarrow H^n(p^{-1}(U), U) \oplus H^n(p^{-1}(V), V) \longrightarrow H^n(p^{-1}(U \cap V), U \cap V)$$

The first object is 0 by the suspension isomorphism, and the last arrow is 0 by compatibility. Therefore, we have an isomorphism

$$H^n(p^{-1}(U \cup V), U \cup V) \cong H^n(p^{-1}(U), U) \oplus H^n(p^{-1}(V), V)$$

So we can paste together the classes as desired. By induction, we obtain a Thom class. □

This notion of compatibility turns out to be the same as our usual notion of orientation: recall that an orientation for a vector bundle $p : E \to B$ is a trivializing cover such that the transition functions φ_{UV} all have positive determinant.

Lemma 2.3. *A trivializing cover is an orientation iff all pairs of intersecting open sets are compatible.*

Proof. Suppose that we have a trivializing open cover. Then for any intersecting elements U and V and $x \in U \cap V$, the following commutes:

$$\begin{array}{ccccccc} & & p^{-1}(U) & \longrightarrow & U \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \nearrow & & & & & \downarrow \varphi_{UV}(x) \\ p^{-1}(x) & & & & & & \\ & \searrow & & & & & \\ & & p^{-1}(V) & \longrightarrow & V \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \end{array}$$

Since $\varphi_{UV}(x)^*$ is the identity if the determinant is positive and negation otherwise, we have that all pairs are compatible iff the trivializing cover determines an orientation. □

Combining these lemmas gives us the following theorem:

Theorem 2.4. *A vector bundle $p : E \to B$ has a Thom class for singular cohomology with $\mathbb{Z}$-coefficients iff it is orientable.*

In general, we say that a Thom class with respect to $H^*(-; R)$ *is* an R-orientation, and call a bundle possessing such a Thom class "R-orientable."

Theorem 2.5. *Let B be a finite CW-complex. Then any vector bundle over B is $\mathbb{Z}/2\mathbb{Z}$ orientable.*

Proof. Note that since $\mathbb{Z}/2\mathbb{Z}$ only has one generator, all trivializations are compatible when working with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Thus, any trivializing cover yields a Thom class as in the proof of Lemma 2.2. □

3 The Cohomology of $\mathbb{CP}^n$

We are now ready to calculate $H^*(\mathbb{CP}^n;\mathbb{Z})$. The tautological vector bundle over $\mathbb{CP}^n$ is a complex vector bundle and therefore oriented; hence it has a Thom class. The total space (less the zero section) E^0 in this case is $\mathbb{C}^{2n}\setminus\{0\}$, for which the cohomology $H^k(E^0)$ vanishes in degree $k < 2n-1$. Hence the Gysin sequence tells us that the map $H^k(\mathbb{CP}^n;\mathbb{Z}) \to H^{k+2}(\mathbb{CP}^n;\mathbb{Z})$ given by $c \mapsto c \smile e$ is an isomorphism in degrees $k < 2n$, where e is the Euler class. Consequently, we obtain

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[e]/e^{n+1}$$

Where e is degree 2. In the infinite case, $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[e]$.

4 The Cohomology of $\mathbb{RP}^n$

$H^*(\mathbb{RP}^n)$ will require slightly more care. Again we consider the tautological bundle over $\mathbb{RP}^n$ - but this is not orientable. However, like any bundle, it is $\mathbb{Z}/2$ orientable. By the same method as the last section, we obtain that the $\mathbb{Z}/2\mathbb{Z}$-cohomology of $\mathbb{RP}^n$ is $\mathbb{Z}/2\mathbb{Z}[e]/e^{n+1}$, where e is degree 1. In the infinite case, $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[e]$.

Now we will use this in order to deduce the the cohomology of $\mathbb{RP}^n$ with $\mathbb{Z}$ coefficients. For this, we will need to understand the cellular chain complex: there is one cell in each degree, and the attaching map $S^n \to \mathbb{RP}^n$ is the standard double cover. The composition $S^n \to \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$ factors through the pinched sphere $S^n \vee S^n$, with the induced map being $id_n \vee A_n$, where A_n is the antipodal map. In a diagram,

$$\begin{array}{ccccc} S^n & \longrightarrow & \mathbb{RP}^n & \longrightarrow & S^n \\ & \overset{\mu}{\searrow} & & \nearrow_{id_n \vee A_n} & \\ & & S^n \vee S^n & & \end{array}$$

With μ being the standard map of the h-cogroup structure on S^n. The map A_n has degree $(-1)^{n+1}$, meaning that our resulting cellular complex looks like:

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots$$

From which we can calculate the cohomology groups of $\mathbb{RP}^n$. To calculate the ring structure, note that the map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ induces a map of chain complexes:

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \cdots \end{array}$$

Because the induced map $H^*(\mathbb{RP}^n;\mathbb{Z}) \to H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ is injective in all positive degrees (except n, when n is odd), the cup product in the latter determines the cup product in the former. For even dimensional projective space, we get

$$H^*(\mathbb{RP}^{2n};\mathbb{Z}) \cong \mathbb{Z}[x]/(2x, x^{n+1})$$

Where the degree of x is 2. For odd dimensional, we have to add in another variable for the top degree generator, giving us

$$H^*(\mathbb{RP}^{2n+1};\mathbb{Z}) \cong \mathbb{Z}[x,y]/(2x, x^{n+1}, xy, y^2)$$

Where the degree of x is again 2 and the degree of y is $2n+1$. In the infinite case,

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]/(2x)$$

With x degree 2.