Embedded Contact Homology of Prequantization Bundles

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WHVSS, May 2020

A contact structure is a maximally nonintegrable hyperplane field.

\[ \xi = \ker(dz - ydx) \]
Contact structures

Definition

A contact structure is a maximally nonintegrable hyperplane field.

\[ \xi = \ker(dz - ydx) \]

The kernel of a 1-form \( \lambda \) on \( Y^{2n-1} \) is a contact structure whenever

- \( \lambda \wedge (d\lambda)^{n-1} \) is a volume form \( \iff d\lambda|_\xi \) is nondegenerate.
Reeb vector fields

**Definition**

The Reeb vector field $R$ on $(Y, \lambda)$ is uniquely determined by

- $\lambda(R) = 1$,
- $d\lambda(R, \cdot) = 0$.

The **Reeb flow**, $\varphi_t : Y \to Y$ is defined by $\frac{d}{dt} \varphi_t(x) = R(\varphi_t(x))$.

A closed **Reeb orbit** (modulo reparametrization) satisfies

\[
\gamma : \mathbb{R}/T\mathbb{Z} \to Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \quad (0.1)
\]

and is **embedded** whenever (0.1) is injective.
Given an embedded Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$, the linearized flow along $\gamma$ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$ is called the linearized return map.

If 1 is not an eigenvalue of $d\varphi_T$ then $\gamma$ is nondegenerate.

Nondegenerate orbits are either elliptic or hyperbolic according to whether $d\varphi_T$ has eigenvalues on $S^1$ or real eigenvalues.

$\lambda$ is nondegenerate if all Reeb orbits associated to $\lambda$ are nondegenerate.
Reeb orbits on $S^3$

$S^3 := \{(u, v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \lambda = \frac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv)$.

The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu \frac{\partial}{\partial u} - i\bar{u} \frac{\partial}{\partial \bar{u}} + iv \frac{\partial}{\partial v} - i\bar{v} \frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u, v) = (e^{it}u, e^{it}v)$.

Patrick Massot

Niles Johnson, $S^3/S^1 = S^2$
A video of the Hopf fibration

The Hopf Fibration

Niles Johnson
http://www.nilesjohnson.net
Theorem (Boothby-Wang construction ’58)

Let \((\Sigma_g, \omega)\) be a Riemann surface and \(e\) a negative class in
\(H_2(\Sigma_g; \mathbb{Z})\). Let \(p : Y \rightarrow \Sigma_g\) be the principal
\(S^1\)-bundle with Euler class \(e\). Then there is a connection
1-form \(\lambda\) on \(Y\) whose Reeb vector field \(R\) is tangent to the \(S^1\)-action.

\((Y, \lambda)\) is the **prequantization bundle** over \((\Sigma_g, \omega)\).

- The Reeb orbits of \(R\) are the \(S^1\)-fibers of this bundle.
- The Reeb orbits of \(R\) are degenerate.

\[d\lambda = p^*\omega\]

\[p_*\xi = T\Sigma_g\]
Use a Morse-Smale $H : \Sigma \to \mathbb{R}$, $|H|_{C^2} < 1$ to perturb $\lambda$:

$$\lambda_\varepsilon := (1 + \varepsilon p^* H) \lambda$$

The perturbed Reeb vector field is

$$R_\varepsilon = \frac{R}{1 + \varepsilon p^* H} + \varepsilon \tilde{X}_H \quad \frac{(1 + \varepsilon p^* H)^2}{(1 + \varepsilon p^* H)^2}$$

where $\tilde{X}_H$ is the horizontal lift of $X_H$ to $\xi$. If $p \in \text{Crit}(H)$ then $X_H(p) = 0$.

The **action** of a closed orbit $\gamma$ is $A(\gamma) := \int_\gamma \lambda_\varepsilon$.

Fix $L > 0$. $\exists \varepsilon > 0$ such that if $\gamma$ is an orbit of $R_\varepsilon$ and

- if $A(\gamma) < L$ then $\gamma$ is nondegenerate and projects to $p \in \text{Crit}(H)$;
- if $A(\gamma) > L$ then $\gamma$ loops around the tori above the orbits of $X_H$, or is a larger iterate of a fiber above $p \in \text{Crit}(H)$.
Recall
\[ R_\varepsilon = \frac{R}{1 + \varepsilon p^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon p^* H)^2} \]

Denote the \( k \)-fold cover projecting to \( p \in \text{Crit}(H) \) by \( \gamma_p^k \).

We have
\[ CZ_\tau(\gamma_p^k) = RS_\tau(\text{fiber}^k) - \frac{\dim(\Sigma)}{2} + \text{ind}_p(H). \]

Using the constant trivialization of \( \xi = p^* T\Sigma \), \( RS_\tau(\text{fiber}^k) = 0 \).

Thus
\[ CZ_\tau(\gamma_p^k) = \text{ind}_p(H) - 1. \]
Recall

\[ CZ_{\tau}(\gamma^k_p) = \text{ind}_p(H) - 1 \]

Only positive hyperbolic orbits have even CZ.

If \( \text{ind}_p(H) = 1 \) then \( \gamma_p \) is positive hyperbolic.

Since \( p \) is a bundle, all linearized return maps are close to \( \text{Id} \).

Hence no negative hyperbolic orbits.

If \( \text{ind}_p(H) = 0, 2 \) then \( \gamma_p \) is elliptic.

Assume \( H \) is perfect. Denote

- the index zero elliptic orbit by \( e_- \)
- the index two elliptic orbit by \( e_+ \),
- the hyperbolic orbits by \( h_1, \ldots, h_{2g} \).
Embedded contact homology (ECH) is a Floer theory for closed $(Y^3, \lambda)$ and $\Gamma \in H_1(Y; \mathbb{Z})$.

For nondegenerate $\lambda$, the chain complex $ECC_*(Y, \lambda, \Gamma, J)$ is generated as a $\mathbb{Z}_2$ vector space by orbit sets $\alpha = \{ (\alpha_i, m_i) \}$, which are finite sets for which:

- $\alpha_i$ is an embedded Reeb orbit
- $m_i \in \mathbb{Z}_{>0}$
- $\sum_i m_i[\alpha_i] = \Gamma$
- If $\alpha_i$ is hyperbolic, $m_i = 1$.

The grading $*$ comes from the relative ECH index $I(\alpha, \beta)$, a combination of $c_1(\text{ker } \lambda)$, $CZ(\alpha_i^k)$, $CZ(\beta_j^k)$, and the relative self-intersection.
A \( \lambda \)-compatible almost complex structure is a complex structure \( J \) on \( T(\mathbb{R} \times Y) \), for which:

- \( J \) is \( \mathbb{R} \)-invariant
- \( J\xi = \xi \), positively with respect to \( d\lambda \)
- \( J(\partial_s) = R \), where \( s \) denotes the \( \mathbb{R} \) coordinate

\( \langle \partial^{ECH} \alpha, \beta \rangle \) counts **currents**, disjoint unions of \( J \)-holomorphic curves

\[
u : (\hat{\Sigma}, j) \to (\mathbb{R} \times Y, J), \quad du \circ j = J \circ du\]

which are asymptotically cylindrical to orbit sets \( \alpha \) and \( \beta \) at \( \pm \infty \).

For generic \( J \),

**ECH index one yields somewhere injective.**

-Hutchings’ Haiku
Theorem (Hutchings-Taubes ’09)

$$(\partial^{ECH})^2 = 0, \text{ so } (ECC_*(Y, \lambda, \Gamma, J), \partial^{ECH}) \text{ is a chain complex.}$$

Theorem (Taubes, Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda)

*The homology depends only on* $(Y, \ker \lambda, \Gamma)$.

We denote the homology by $ECH_*(Y, \ker \lambda, \Gamma)$.

*Dee squared is zero; obstruction bundle gluing is complicated.*

-Hutchings-Taubes’ Haiku
Theorem (Nelson-Weiler, 90%)

Let \((Y, \xi = \ker \lambda)\) be a prequantization bundle over \((\Sigma_g, \omega)\). Then

\[
\bigoplus_{\Gamma \in H_1(Y; \mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)
\]

Inspired by the 2011 PhD thesis of Farris.

1. The critical points of a perfect \(H\) form a basis for \(H_*(\Sigma_g; \mathbb{Z}_2)\). The generators of \(ECH\) are of the form \(e_-^{m-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m+}\) where \(m_i = 0, 1\), so correspond to a basis for \(\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)\).

2. We will prove \(\partial^{ECH}\) only counts cylinders corresponding to Morse flows on \(\Sigma_g\), therefore \(\partial^{ECH}(e_-^{m-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m+})\) is a sum over all ways to apply \(\partial^{Morse}\) to \(h_i\) or \(e_+\).
Our favorite fibration on $S^3$

**Example ($S^3, \lambda$)**

The ECH of $S^3$ is the $\mathbb{Z}_2$-vector space generated by terms $e_-^m e_+^m$, where $|e_-| = 2, |e_+| = 4$. Note that $\ast$ is not the grading on $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$, since $|e_-^2| = 6$.

The fibers above the critical points of the height function on $S^2$ represent $e_\pm$.

We have $\partial^{ECH} = 0$ because $\partial^{Morse} = 0$. 

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Embedded Contact Homology of Prequantization Bundles
Lens spaces $L(k, 1)$

$L(k, 1)$ is the total space of the prequantization bundle with Euler number $-k$ on $S^2$.

**Corollary (Nelson-Weiler, 95%)**

With its prequantization contact structure $\xi_k$,

$$ECH_*(L(k, 1), \xi_k, \Gamma) \cong \begin{cases} 
\mathbb{Z}_2 & \text{if } * \in 2\mathbb{Z}_{\geq 0} \\
0 & \text{else}
\end{cases}$$

for all $\Gamma \in H_1(L(k, 1); \mathbb{Z})$. 
Finer points of the isomorphism

Fix a negative Euler class $e$. For $\Gamma \in \{0, \ldots, -e - 1\}$,

$$ECH_* (Y, \xi, \Gamma) = \bigoplus_{n \in \mathbb{Z} \geq 0} \Lambda^{\Gamma - ne} (H_*(\Sigma_g; \mathbb{Z}_2))$$

**Proposition (Nelson-Weiler)**

Let $\alpha = e_-^m h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^m$ and let $\beta = e_-^n h_1^{n_1} \cdots h_{2g}^{n_{2g}} e_+^n$.

Let $N = n_- + n_+ + \sum_j n_j$ and $m = \frac{(m_- + m_+ + \sum_i m_i) - N}{-e}$. Then

$$I(\alpha, \beta) = (2 - 2g)m - m^2 e + 2mN + m_+ - m_- - n_+ + n_-$$

Using this formula, we obtain

$$I(e_+^{N+e}, e_-^N) = 2g - 2$$
ECH∗(Y, ξ, 0) for g = 2, e = −1

Recall I(eN⁺, eN⁻) = 2g − 2. Set *(α) = I(α, ∅).

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Embedded Contact Homology of Prequantization Bundles
Outline of proof

**Theorem (Nelson-Weiler, 90%)**

Let \((Y, \xi = \ker \lambda)\) be a prequantization bundle over \((\Sigma_g, \omega)\). Then

\[
\bigoplus_{\Gamma \in H_1(Y; \mathbb{Z})} ECH_* (Y, \xi, \Gamma) \cong \mathbb{Z}_2 \wedge^* H_* (\Sigma_g; \mathbb{Z}_2)
\]

1. There exists \(\varepsilon > 0\) so that the generators of \(ECC^L_*(Y, \lambda_\varepsilon, J)\) consist solely of orbits which are fibers over critical points.

2. Prove that \(\partial^{ECH,L}\) only counts cylinders which are the union of fibers over Morse flow lines in \(\Sigma\).

3. Finish with a direct limit argument, sending \(\varepsilon \to 0\) and \(L \to \infty\), in addition to the isomorphism with Seiberg-Witten.
Pseudoholomorphic Cylinders

- Pseudoholomorphic cylinders correspond to Floer trajectories on \( \Sigma_g \) (Moreno, Siefring)
- Floer trajectories on \( \Sigma_g \) correspond to Morse flows (Floer, Salamon-Zehnder)
- Cylinder counts permit use of fiberwise \( S^1 \)-invariant \( J \), even for multiply covered curves, by automatic transversality (Wendl)

Theorem (N. 2017)

The cylindrical contact homology chain complex of a prequantization bundle over \( \Sigma_g \) is generated by infinitely many copies of the Morse complex of \( \Sigma_g \), and on each copy the cylindrical differential agrees with the Morse differential.
Can count cylinders using the complex structure $J_{\Sigma_g} = p^* j_{\Sigma_g}$, the $S^1$-invariant lift of $j_{\Sigma_g}$.

(YAY! Automatic transversality!)

$J_{\Sigma_g}$-holomorphic cylinders correspond to Morse trajectories on $\Sigma_g$.

Cannot use $J_{\Sigma_g}$ for higher genus curves!

$J_{\Sigma_g}$ cannot be independently perturbed at the intersection points of $\pi_Y u$ with a given $S^1$-orbit by an $S^1$-invariant perturbation.

(YIKES! $J_{\Sigma_g}$ is not typically regular!)

There will always be a regular $J$ for moduli spaces of higher genus curves, but we cannot assume $J$ is $S^1$-invariant.

(CURVE COUNTING NO LONGER OBVIOUS...)
Forsake $J_{\Sigma_g}$ for an $S^1$-invariant domain dependent perturbation,

$$\{ J_{\Sigma_g}^z \}_{z \in \hat{\Sigma}}$$

- Akin to time-dependence in Hamiltonian Floer theory.
- Implicit function theorem relates counts of nearby moduli spaces.

**Higher genus curves and multiply covered cylinders do not contribute to** $\partial^{ECH}$

- Transversality guarantees index 1 holomorphic curves do not exist unless they are fixed by the $S^1$-action.
- Otherwise the curve lives in a moduli space of dimension $\geq 2$.
- But $\langle \partial^{ECH} \alpha, \beta \rangle$ only counts curves where $\pi_Y \circ u$ is isolated.
- So we only count cylinders projecting to Floer trajectories.

**Remaining issue** (modulo direct limits):

- Hutchings set up ECH with a domain independent $J$...
Consider \( \{ \tilde{J}_t \}_{t \in [0,1]} \) a family of \( S^1 \)-invariant domain dependent almost complex structures in \( \mathbb{R} \times Y \),

\[
\begin{align*}
\tilde{J}_0 &:= \left\{ J^Z_{\Sigma g} \right\}_{z \in \dot{\Sigma}} \\
\tilde{J}_1 &:= J \in \mathcal{J}^{\text{reg}}(Y, \lambda).
\end{align*}
\]

**Lemma**

For generic paths, the moduli space \( \mathcal{M}_t = \mathcal{M}(\alpha, \beta, \tilde{J}_t) \) is cut out transversely save for a discrete number of times \( t_0, \ldots, t_\ell \in (0,1) \). For each such \( t_i \), \( \partial^{ECH} \) can change either by

- creation/destruction of a pair of oppositely signed curves;
- an “ECH handleslide.”

In either case, the homology is unaffected. PHEW!
Handleslides do not impact curve counts

At a handleslide $t_i$, $\{ C_k \mid \text{ind}_{\text{Fred}}(C_k) = 1 \}$ breaks into a building with:

- an index 1 curve $C_1$ at top (or bottom)
- branched covers $C$ of $\gamma \times \mathbb{R}$ with $\text{ind}_{\text{Fred}}(C) = 0$
- an index 0 curve $C_0$ at bottom (or top)

*Branched covers cannot appear as the top-most or bottom-most level.*

(Hutchings - N ‘16, Cristofaro-Gardiner - Hutchings - Zhang)

*Hooray! We can invoke obstruction bundle gluing...*

\[
\# M(\alpha, \beta, \mathcal{J}_{t_i+\epsilon}) = \# M(\alpha, \gamma, \mathcal{J}_{t_i-\epsilon}) + \# G(C_1, C_0) \cdot \# M(\gamma, \beta, \mathcal{J}_{t_i}),
\]

OBG gives a combinatorial formula for $\# G \in \mathbb{Z}$, based on
the partitions at $-\infty$ ends of $C_1$,
the partitions at $+\infty$ ends of $C_0$.
No need to explicitly compute $\# G$ as inductively $\# M(\gamma, \beta, \mathcal{J}_{t_i}) = 0$!
Filtrations and computations

There is no geometric Morse-Bott ECH.

Denote by $ECH^L_*(Y, \lambda_\varepsilon, \Gamma)$ the homology of the chain complex of ECH generators with action $\leq L$. (It’s independent of $J$.)

Hutchings-Taubes ’13: Cobordism and inclusion maps give us

\[
\begin{array}{ccc}
ECH^L_*(Y, \lambda_\varepsilon, \Gamma) & \longrightarrow & ECH^L_*(Y, \lambda_\varepsilon', \Gamma) \\
\downarrow & & \downarrow \\
ECH^{L'}_*(Y, \lambda_\varepsilon, \Gamma) & \longrightarrow & ECH^{L'}_*(Y, \lambda_\varepsilon', \Gamma)
\end{array}
\]

which commute, for $\varepsilon' < \varepsilon, L' > L$.

We can now compute

\[
\lim_{\varepsilon \to 0, L \to \infty} ECH^L_*(Y, \lambda_\varepsilon, \Gamma, J) \cong \mathbb{Z}_2 \wedge^* H_*(\Sigma_g; \mathbb{Z}_2). \quad (0.2)
\]

That the LHS of (0.2) is $ECH_*(Y, \xi, \Gamma)$ uses a similar filtration on Seiberg-Witten Floer homology from Hutchings-Taubes.
Future work: \( U \) map

There is a degree \(-2\) map

\[
U : ECC_*(Y, \xi, \Gamma) \to ECC_{*-2}(Y, \xi, \Gamma)
\]

which counts \( J \)-holomorphic curves passing through a base point. \( U \) is equivalent to the \( U \) maps on Seiberg-Witten and Heegaard Floer homologies.

In the case of prequantization bundles, we expect \( U \) to count index 2 Morse flow lines and sections of the \( \mathbb{C}^* \) bundle \( \mathbb{R} \times Y \to \Sigma_g \).

\( U \) is Useful:

- Find index 2 holomorphic curves, since \( U \) is an invariant;
- **ECH capacities**, which obstruct symplectic embeddings;
- Proving stabilization results.
Future work: stabilization

From Seiberg-Witten and Heegaard Floer homologies we know $U$ is an isomorphism if $*$ is large enough. Therefore:

**Theorem (Nelson-Weiler, 90%)**

If $e = -1$ and $g > 1$, then for $*$ large enough,

$$ECH_* (Y , \xi ) \cong \mathbb{Z}_2^{2g - 1}.$$  

and $U$ is an isomorphism.

We expect to prove this theorem entirely in ECH once we can compute the $U$ map.
Future work: surface dynamics

Proposition (Colin-Honda ’13)

If $\phi \in \text{Mod}(\Sigma_g)$ is periodic, then $(Y, \xi)$ is supported by an open book decomposition with page $\Sigma_g$ and monodromy $\phi$ and is a Seifert fiber space over the orbifold $\Sigma_g/\phi$. There is a contact form for $\xi$ whose Reeb vector field is tangent to the fibers.

We will generalize our prequantization methods to circle bundles over orbifolds to understand the dynamics of symplectomorphisms which are freely homotopic to $\phi$, extending the Calabi invariant bounds in Weiler’s thesis to genus 0 open books.
Thanks

Ages 4-7

The Learning Company

Math Rabbit

Contact Homology

compactness!

transversality!

and other fun with pseudoholomorphic curves!!

Each activity keeps kids busy for decades!

\[ E(u) := \sup_{\phi \in \mathcal{I}} \int_{\Sigma} u^* d\alpha. \]