Math 4540 Final Project

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1 Introduction

Throughout our study of Differential Geometry, we've avoided the formal study of manifolds and differential forms despite its similarity with the content we encounter. Here, I will be exploring the connection between our course material with concepts like tangent bundles, manifolds, cotangent bundles and their connection to high level physics.

2 Manifolds and Tangent Bundles

Manifolds are topological spaces that are locally Euclidean near each point. We have seen many two dimensional manifolds through our study of surfaces which are defined by an Atlas or a collection of Surface Patches.

The tangent bundle TV of a manifold V is itself a manifold given by the disjoint union of tangent spaces of V.

$$TV = \bigcup_{x \in V} \{x\} \times T_x V$$

We can only visualize tangent bundles on one dimensional manifolds as the dimension of the tangent bundle of a manifold of dimension n is 2n. So it is impossible for us to visualize the tangent bundle of any surfaces as that would lead to a four dimensional tangent bundle. A trivial example of a tangent bundle is of S^1 which is an infinite cylinder around the circle.

$$TR^n = \bigcup_{x \in R^n} T_x R^n = R^n \times R^n$$

Above we see the form of the tangent bundle for R^n which gives $R^n \times R^n$. When the tangent bundle to a manifold V is in the form of $V \times R^n$, we say the tangent bundle is trivial and the manifold is parallelizable. As is the case for S^1 and R^n.

3 Derivative of Smooth Map as a Map between Tangent Bundles

There is an obvious but important projection from the tangent bundle TV to V which is given by $\pi(x,v) = x$ where $x \in V$ and $v \in TV$. Which is useful in allowing us to use a single letter like v to represent and element of TV.

We define the differential of a smooth map between two manifolds V and M and follows:

$$\phi : V \to M$$

$$d\phi_x = T_xV \to T_{\phi(x)}M$$

We can now use this to create a bundle map between the tangent bundles of both manifolds.

$$\begin{array}{ccc} TV & \xrightarrow{d\phi} & TM \\ \Big\downarrow{\scriptstyle \pi_v} & & \Big\downarrow{\scriptstyle \pi_M} \\ V & \xrightarrow{\phi} & M \end{array}$$

As we can see, the differential of a smooth map is used to go from one tangent bundle to another while the projection π just gets us back to our manifold.

4 Section of a Tangent Bundle and the First Fundamental Form

A smooth map of all of a manifolds tangent vectors to each point is called a vector field W. Specifically: $W : V \to TV$. This is called a section on V and we call W a section of the tangent bundle on V. Each element of the vector space $W(x) = (x, W_x)$ is thus a tangent vector of V. This is useful because it allows us to use the linear properties of Vector Fields to see:

$$(W + P)_x = W_x + P_x$$

This statement says that addition of two tangent vectors from two sections of the tangent bundle of V is simply the tangent vector under W + the tangent vector under P.

Recall that the First Fundamental Form of a surface (2-dimensional manifold) is the inner product of two tangent vectors. From it we get the metric tensor whose components are two tangent vectors dotted into each other.

$$g_{ij} = T_iV \cdot T_jV$$

Where both T_iV and T_jV are elements of the tangent bundle TV. I hope you can now see the importance of tangent bundles as we've worked with them throughout the class, just not explicitly.

5 Connecting it all together: Cotangent Bundle

We can define the Cotangent Space as a vector space which takes points from T_xV onto the scalar field which the manifold lives in (often R^n). This means that every element $\alpha \in T_x^*V$ is a map $\alpha : T_xV \to R^n$. To get a better intuition, we can think of an example of the function temperature on our planet. The cotangent bundle will give you the change in the temperature for a given direction. Now we can see how the cotangent bundle takes functions and turn them into scalars.

We can define a bundle map $\tilde{g} : T_xV \to T_x^*V$. Which, for reasons that you can find here, is an isomorphism. The First Fundamental Form is the matrix of the isomorphism $\tilde{g}$ as it takes $g_x : T_xV \times T_xV \to R$. The gradient can similarly be defined in this way:

$$\nabla f = \tilde{g}^{-1}(df)$$

. We can then see that if $t \in T_xV$,

$$< \nabla f, t >_g = \tilde{g}(\nabla f)(t) = df(t) = tf$$

. So we get

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

where g^{ij} is the metric tensor.

6 Visualization: Least Action Principle

Taking a step back and focusing on an application to the course, we will explore the least action principle from physics. The least action principle states that the trajectory of any particle with defined end points will always minimize what we call the action S. S is a functional equal to the integral of the Lagrangian which is Kinetic Energy minus Potential Energy of the system. After extremizing S, one is greeted with the Euler-Lagrange Equation(s) which allows one to find the equations of motion for the system. This is similar to how geodesics minimize the length function because they result in the same minimization of space/time. Einstein showed that one can find the geodesic equations by using the principle of least action on spacetime.

7 Visualization: Minkowski Metric

Minkowski space is a 4-dimensional differential manifold made up of the Euclidean space we're used to plus a time dimension. This manifold is extremely applicable to general and special relativity as the spacetime interval between any two events is independent of the inertia reference frame that the two events took place in. The metric tensor, or First Fundamental Form for us, of Minkowski space has eigenvalues with signature -,+,+,+. Recall that a metric is what we use to measure distance in different manifolds. One can clearly see the importance of finding the metric for Minkowski space as it governs all interactions. In general relativity, gravity is not a force but a consequence of curved spacetime. So all objects only under the direct influence of gravity follow a geodesic in Minkowski space.

8 Connection and Gauge Theory

We discussed connections(covariant derivatives) in chapter 7 regarding parallel transport. They allowed us to transport vectors along the manifold such that they remain parallel with respect to the connection. We can define a connection on the tangent bundle of V as an affine connection which is a connection that connects nearby tangent spaces. This is a special case because it allows us to look at affine geodesics which are defined under this condition $\nabla_{\gamma^{\cdot}(t)}\gamma(t) = 0$. Gauge theory is the study of these vector bundles and their properties. It's research in math an physics has given rise to new differential equations solving the connections between bundles. Gauge theory in physics is closely related as it's a field theory that studies when the Lagrangian is invariant under transformations according to Lie groups.

9 Conclusion

I hope you've been able to see the deep connections that the content we've learned over the course of the semester has with other topics in physics and math. From vector bundles to general relativity, there are a lot of concepts that we still have yet to learn but now have the tools to start!