An overview of Triangulation and associated homologies

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Introduction

Theorem 13.4.3 in Pressley, which states that "every compact surface has a triangulation with finitely many polygons" is a version of the Triangulation Theorem for surfaces. It turns out that this is the two dimensional case of the infamous Triangulation Conjecture, which proposes that a geometric space can be decomposed into smaller pieces, or triangulated. This will be the topic for my final-mini project.

The outline of this project is as follows: I will first discuss a brief history of the Triangulation Conjecture (and how it was recently disproved), give a brief, high-level outline of the proof of the Triangulation Conjecture in two dimensions, as well as give a brief overview of how it relates to homology theory. I will also calculate the homologies of the 2-sphere as the second part of my project.

The History of Triangulation

Can every topological manifold be triangulated? This question is called the Triangulation Conjecture, and has puzzled mathematicians until the late 20th century. It has long been well-known that any two-dimensional surface or 3-dimensional space can be subdivided into smaller pieces, ie. triangulated, but higher dimensions were not so clear. In 1980, Casson gave counterexamples in dimension 4, but it was not until much more recently, in 2015, that a proof that the Triangulation Conjecture is false for every $n \geq 5$, was found by Manolescu. Nevertheless, triangulation maintains an exceedingly useful tool for two-dimensional manifolds and three-dimensional spaces by giving a concrete way of representing and visualizing spaces that may be otherwise difficult to analyze.

Sketch of a Proof of the two-dimensional case

Recall that the definition of a triangulation of surface S is as follows:

Definition 1. Let S be a surface, with atlas consisting of the patches $\sigma_i : U_i \to \mathbb{R}^3$. A *triangulation* is a collection of curvilinear polygons, each of which is contained, together with its interior, in one of the $\sigma_i(U_i)$, such that:
(i) Every point of S is in at least one of the curvilinear polygons.
(ii) Two curvilinear polygons are either disjoint, or their intersection is a common edge or a common vertex.
(iii) Each edge is an edge of exactly two polygons.

And the theorem we are trying to prove is:

Theorem 1. Every compact surface has a triangulation with finitely many polygons.

This is stated without proof in Pressley. Here, we also do not prove it in entirety but give a brief outline of what the proof might look like. To do so, we introduce simplices and simplicial compexes, which informally is a generalization of a triangle or tetrahedra to arbitrary dimensions; the k-simplex is the convex hull of its k+1 vertices; thus a 0-simplex is a point, a 1 simplex is a line segment, a 2 simplex a triangle, a 3-simplex a tetrahedron, and an n-simplex an n+1-cell.

Definition 2. A *Euclidean simplicial complex* is a collection K of simplices in $\mathbb{R}^n$ satisfying the following conditions:
(1) If $\sigma \in K$, then every face of σ is in K.
(2) The intersection of any two simplices in K is either empty or a face of each.
(3) Every point in a simplex of K has a neighborhood that intersects finitely many simplices of K.

A triangulation is simply a particular homeomorphism between a topological space and a Euclidean simplicial complex, and it is evident that this definition is equivalent to the two-dimensional triangulation definition given in Pressley. Thus we will prove an equivalent statement to Theorem 1, that "Every surface S is homeomorphic to a triangulated surface".

Below is a brief outline of the proof. It is adapted primarily from the proof by C.Thomassen, and details can be found in [1]

Proof.
The basic idea behind this proof is to cover S with disks and show that each disk can be triangulated compatibly. We need to know the following:

Schonflies Theorem: A simple closed curve J separates $\mathbb{R}^2$ into two regions, and there is a homeomorphism of R^2 to itself such that J is mapped to a circle. (In other words, an embedding of a circle in $\mathbb{R}^2$ can be extended to the embedding of a disk in $\mathbb{R}^2$).

We assume that the interior of a convex polygon can be triangulated so we just have to show S is homemorphic to a surface with a 2-cell embedding.

Covering S: For each point p on S, let $D(p)$ be a disc on the plane, homeomorphic to some neighborhood of p on S. In $D(p)$, let there be two quadrilaterals $Q_1(p)$ and $Q_2(p)$ such that p is an element of the interior of $Q_1(p)$, which is in the interior of $Q_2(p)$. We have some finite number of points $p_1...p_n$ (due to compactness of S, it has a finite subcover) such that $S = \bigcup_{i=1}^{n} int(Q_1(p_i))$. In other words, there is a sequence of quadrilaterals including the sequence of points p_i whose union of interiors is equal to S. We want to show that $Q_1(p_1)...Q_1(p_n)$ can be chosen to form a 2-cell embedding of S.

Triangulating each disk: From here the details of the proof are mostly omitted, and can be found in [1] if needed. The first step is to show that $Q_1(p_1)...Q_1(p_n)$ can be chosen so that any two quadrilaterals have only a finite number of points in common on S. This is done inductively on k, and the basic idea is do redefine the boundary of a new $Q_1(p_n)$ as a graph on S which is homeomorphic to a connected graph which the proof constructs on the plane such that there are finite intersections. Then by a variant of the Schonflies Theorem, this homeomorphism can be extended to the entirety of the plane, thus redefining a $Q_1(p_n)$ which only has finite intersections with the other quadrilaterals.

Then there are finitely many segments in each $Q_2(p_k)$ and these form a 2-connected plane graph. Consider then, some simply closed polygonal curves in each $Q_i2(p_i)$, and draw polygons C' whose corners correspond to vertices of each C, and it then the union of these polygons C' form. It turns out we have (by restricting a isomorphism between graphs which is discussed more thoroughly in [1]) a homeomorphism between C, and C', and by the Schonflies Theorem, we can extend to a homeomorphism of $int(C)$ to $int(C')$, thus S is homeomorphic to the surface triangulated by the C' (recall we explicitly constructed polygons C' so that their vertices correspond to vertices of the curves of C). $\square$

Triangulation in Simplicial Homology

Often spaces of higher dimension can be difficult to analyze, and we may want to use tools from homological algebra (with which can be easier to manipulate and deduce properties of the associated spaces from) to gain more insight of these topological spaces. In this section I will briefly explain some basic concepts and tools in simplicial homological algebra, and show how these may be used to gain more insight into our topological spaces and shapes. Part 2 of this project will be an examples of finding Betti numbers of S^2 (and the associated simplicial complexes through the introduced methods).

Suppose we want to find out the number of "holes" in a space we are interested in. First, what do we mean by "holes"? It appears to depend on what dimensional hole we are referring to. For instance, a circle should have 1 hole, its interior. But what about a 2-sphere, or even worse, a torus? Do we say it doesn't have any holes (a single cut would slice it into two pieces), or do we count its empty cavity as a hole? It turns out that we need to define holes on different dimensions, which is closely related to the concept of Betti numbers. Informally, the nth Betti number is the number of (n+1)-dimensional holes a space has. Formally, the nth Betti number is the rank of the nth "homology group" of a topological space, but we need to define homology group, which we will now do.

First we define chain complexes; we will eventually construct a type of chain complex from simplicial complexes of the spaces we are interested in:

Definition 3. A *chain complex* is a sequence of abelian groups or modules connected by homomorphisms (called differentials and denoted $d_n : X_n \to X_{n-1}$), such that $d^2 = 0$.

A chain complex looks like:

$$... \to X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \to ...$$

Definition 4. Let K be a simplicial complex of dimension n with p-simplexes σ_p^i. The *pth chain group* of K, denoted $C_p(K)$, is the free abelian group on set $\{\sigma_p^1, ...\sigma_p^m\}$ (an element of this group, a linear combination of some p-simplexes σ, is called a *p-chain).* The boundary homomorphism d_p : $C_p(K) \to C_{p-1}(K), d_p(\sigma) = (-1)^j \sum_{j=0}^{p}[u_0, ...u_{j-1}, u_{j+1}, u_p]$ gives the alternating sum of the (p-1)-simplexes corresponding to the faces of each p-simplex (and recall that each p-chain is a linear combination of p-simplexes).

We can verify that the differentials form a chain complex which we call the *chain complex of* K , see [2] for details. Now we can define homology groups:

Definition 5. The p*th homology group* is $H_p = C_p(K)/C_{p-1}(K)$. The *p-th Betti number* is the rank of this group, $\beta_p = rank(H_p)$.

From this, we see that we can actually recover information (from the Betti numbers) from the chain complexes associated with the simplicial complex built from triangulating surfaces. Since we have a way of forming a chain complex from a simplicial complex, and we can triangulate any surface, it means we can construct the corresponding chain complex for any surface. From this, we can compute the *pth* homology group, from which we can calculate the Betti numbers, which gives us information about the number of holes in the surface. Another interesting note is that the Euler characteristic of a surface, which we talked about in class, is simply the alternating sum of the Betti numbers of that surface.

Part 2: Example of Computing Homology Groups: S^2

Suppose we want to compute the Betti numbers and the homology groups of the 2-sphere, S^2. In the previous section, we have outlined a method to do this:
1) Triangulate the surface, and view this as a simplicial complex K
2) Construct a chain complex for K
3) Compute the homology groups and deduce Betti numbers

1) Triangulating the surface: This is simple, in the case of S^2. We can simply take the triangulation of the 2-sphere to be an inflated tetrahedron, as discussed in Pressley pg 351. This also happens to be the simplicial complex consisting of the single 3-simplex (a tetrahedron), which makes our calculations easier. We label the vertices of the tetrahedron as A, B, C, and D, and orient the faces such that the edges are oppositely oriented. So we have:
0-simplices: [A], [B], [C], [D]
1-simplices: [AB], [BC], [CA], [AD], [DC], [DB]
2-simplices: [ABC], [ACD], [ADB], [BCD]

2) Constructing the Chain Complex:
The pth chain groups are the free abelian groups with the corresponding simplices as generators:
$C_0 : \mathbb{Z}\langle [A], [B], [C], [D] \rangle$
$C_1 : \mathbb{Z}\langle [AB], [BC], [CA], [AD], [DC], [DB] \rangle$
$C_2 : \mathbb{Z}\langle [ABC], [ABD], [ACD], [BCD] \rangle$

3. Now we calculate the homology groups.

H_0:
Here everything is sent to 0, so $\mathrm{Ker}(d_0) = C_0 = \langle A, B, C, D \rangle$
To get $\mathrm{Im}(d_1)$, we have to take the Span of all the boundaries of the 1-simplices: $\langle B - A, C - B, A - C, D - A, C - D, B - D \rangle$
Thus, $H_0(S^2) = \mathrm{Ker}(d_0)/\mathrm{Im}(d_1) \cong \mathbb{Z}$ (quotienting out these boundary relations identifies all the vertices)

H_1:
To get $\mathrm{Ker}(d_1)$, we want to find the linear combinations of the boundaries (of the 1-simplices) that are equal to 0, so:

$$x_1(B-A)+x_2(C-B)+x_3(A-C)+x_4(D-A)+x_5(C-D)+x_6(B-D) = 0$$

$$A(-x_1+x_3-x_4)+B(x_1-x_2+x_6)+C(x_2-x_3+x_5)+D(x_4-x_5-x_6) = 0$$

This gives us 4 equations, 6 unknown variables, which we can solve using the matrix: $\begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

There are 3 free variables, and the solution has basis $\{(x_1+x_2+x_3), (-x_1-x_2+x_4+x_5), (-x_1+x_4+x_6)\}$, therefore $\mathrm{Ker}(d_1) \cong \mathbb{Z}^3$ To get $\mathrm{Im}(d_2)$, we take the Span of all the boundaries of the 2-simplices: $\langle(-x_1+x_3-x_4), (x_1-x_2+x_6), (x_2-x_3+x_5), (x_4-x_5-x_6)\rangle$. Since any $\mathrm{Ker}(d_1)$ can be written as some $\mathrm{Im}(d_2)$, we get $\mathrm{Ker}(d_1)/\mathrm{Im}(d_2) = H_1(S^2) = 0$.

H_2:

$\mathrm{Ker}(d_2) \cong \mathbb{Z}$ because all the 2-simplices must have equal coefficients in order for their 1-dimensional boundaries to cancel (based on our orientations, from construction), so the basis is one dimensional. $\mathrm{Im}(d_3)$ is 0 since there are no 3-simplices, so $H_2(S^2) \cong \mathbb{Z}$

Thus the Betti numbers are $\beta_0 = 1$, $\beta_1 = 0$, and $\beta_2 = 1$. This is expected, as we have 1 connected component, 0 "2-d holes", and 1 "3-d hole" (the concavity). And also, as expected, the Euler characteristic is 1+ 0 +1 =2 by summing the Betti numbers (V-E+F = 4-6+4 = 2 for the tetrahedron, so they agree).

References

[1] B. Mohar and C. Thomassen. Graphs on Surfaces. John Hopkins University Press. 2001.

[2] P. Giblin, Graphs Surfaces and Homology., 2010.