HYPERBOLIC GEOMETRY

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1. Hyperbolic Models

1.1. **Upper Half-Plane Model.** Euclid's Axioms provided a base for Euclidean geometry, but for many, the fifth axiom seemed to not be as self-evident as the others. The parallel axiom states that given any line L and a point p not on the line, there exists a unique line parallel to L that passes through p. For centuries, mathematicians sought to prove this axiom in terms of the other four; early in the 19th century, mathematicians discovered why these efforts always came up short.[1] The parallel axiom is independent of the other axioms and need not hold to produce a consistent system; if we drop it, one consistent geometry that we can get is hyperbolic geometry.

This geometry crops up when we study the pseudosphere. Parametrizing the pseudosphere as $\tilde{\sigma}(v,w) = (\frac{1}{w}\cos v, \frac{1}{w}\sin v, \sqrt{1-\frac{1}{w^2}} - \cosh^{-1} w)$, we find that geodesics correspond to straight lines and circular arcs intersecting the v axis.[3] We must have $w \neq 0$ to ensure that the first fundamental form is defined. However, considering the square root, we see that the curves are only real if $w \geq 1$—this seems odd considering the geodesics exist for $w \in (0,1)$. This motivates us to study the half-plane identified with $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ (where $z = v + iw$ identifies (v,w)), which is equipped with the same first fundamental form as the pseudosphere, $\frac{dv^2+dw^2}{w^2}$.

This gives us the upper half-plane model of hyperbolic space. Using this model, we can quickly show that the parallel axiom does not hold in $\mathcal{H}$, as expected. We can also show that there is a unique hyperbolic line segment connecting any two points, and we can compute the length of this segment to be $2\tanh^{-1}\frac{|b-a|}{|b-\bar{a}|}$.[1] This is the hyperbolic distance between the two points, $d_{\mathcal{H}}(a,b)$.

The last result that we are able to prove using this model is a formula for the area of a hyperbolic polygon, similar to the formula we have previously computed for spherical triangles. These hyperbolic polygons have edges corresponding to the geodesics of the original pseudosphere, so their edges are all arcs of circles centered on the v-axis and lines perpendicular to the w-axis (with $w > 0$).

Denoting the internal angles of an n-sided hyperbolic polygon with $\alpha_1, \alpha_2, \ldots, \alpha_n$, its hyperbolic area $\mathcal{A}$ is $(n-2)\pi - \alpha_1 - \cdots - \alpha_n$. Interestingly, this area depends only on its angles, not the polygon's size or side lengths. The reason for this will become clear when we discuss isometries.

[1] Here $\bar{z}$ denotes the complex conjugate of z

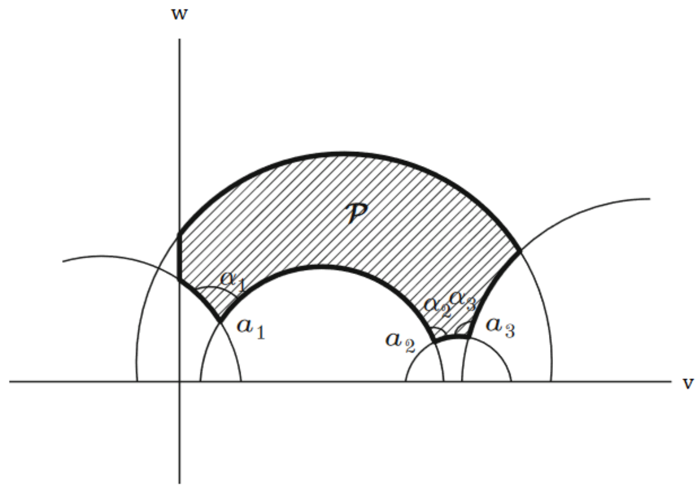

FIGURE 1. An example hyperbolic polygon with some interior angles labeled. Reproduced from page 274 of Pressley.[3]

1.2. **Poincaré Disc Model.** The Poincaré disc model is another way of viewing hyperbolic geometry. We define a transformation $\mathcal{P}(z) = \frac{z-i}{z+i}$ that is well-defined at all points in $\mathcal{H}$. Under this transformation, the half-plane becomes $\mathcal{D} = \{z \in \mathcal{C} \mid |z| < 1\}$. If we equip this with a fundamental form that ensures $\mathcal{P}$ is an isometry, then we have the Poincaré disc model $\mathcal{D}_P$. It turns out that the first fundamental form of $\mathcal{D}_P$ is $\frac{4(du^2+dv^2)}{(1-v^2-w^2)^2}$; since this a multiple of $du^2 + dv^2$, it preserves angles. Also, since $\mathcal{P}$ is an isometry, we can use it to create isometries of $\mathcal{D}_P$; for any isometry F of $\mathcal{H}$, $\mathcal{P} \circ F \circ \mathcal{P}^{-1}$ is an isometry of $\mathcal{D}_P$. We can quickly use this model to translate isometries of $\mathcal{H}$ into isometries of $\mathcal{D}_P$.

First, we note that the hyperbolic lines of $\mathcal{D}_P$ are lines and semicircles that perpendicularly intersect the boundary of $\mathcal{D}$. Thus, inversion in such semicircles and reflection across such lines are isometries of $\mathcal{D}_P$. It is worth noting that because $\mathcal{P}$ is a complicated map, trivial isometries of $\mathcal{H}$ may not correspond to trivial isometries of $\mathcal{D}_P$.[2]

Lastly, the Poincaré disc is a powerful model for proving some fundamental results in hyperbolic trigonometry. The first is the "hyperbolic cosine rule", which states that a hyperbolic triangle with sides A, B, C, and corresponding opposite angles α, β, γ, satisfy $\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma$. Setting γ to be a right angle, this formula gives us the hyperbolic version of the Pythagorean Theorem. Intuitively, it makes sense that the Poincaré map would give us such results; since it is a conformal map with rotational symmetry, it is perfectly suited to proving angle-related results such as these.

[2]One good example of this is rotation of the Poincaré disc, which is a trivial isometry in that model but has very complicated effects for the half-plane model

2. Isometries of $\mathcal{H}$

Earlier we mentioned that the area of a hyperbolic polygon is dependent only on its internal angles. As it turns out, this is due to the isometries of the hyperbolic plane. We know that isometries must take geodesics to geodesics. Thus, isometries of $\mathcal{H}$ must take lines parallel to the imaginary axis to other geodesics, and circular arcs centered on the real axis to vertical lines or other similar arcs. This immediately gives us a few basic isometries:

(1) Translations along the real axis, given by $T_a(z) = z + a$, $a \in \mathbb{R}$.
(2) Reflections across vertical geodesics, given by $R_a(z) = 2a - \bar{z}$, $a \in \mathbb{R}$.
(3) Dilations by a factor of $a > 0$, $D_a(z) = az$.

These transformations are obviously isometries, since they all take lines perpendicular to the v-axis to lines perpendicular to the v-axis, and circular arcs centered on the real axis to other circular arcs centered on the real axis. However, this is not all the possible geodesic transformations; is there an isometry that maps vertical lines to arc geodesics, or vice versa? As it turns out, there is one more isometry that does this.

(4) Inversions in circles with centers on the real axis. Inverting around the circle with center a on the real axis, radius r is given by $\mathcal{I}_{a,r} = a + \frac{r^2}{\bar{z}-a}$.

Similar to how every isometry in $\mathbb{R}^3$ can be decomposed into rotation and translation, we can chain together a finite number of these four elementary isometries to create new isometries of $\mathcal{H}$.

We can show this fourth transformation is an isometry by proving that it preserves the first fundamental form for $a = 0, r = 1$, and then showing that it is a composition of $T_a \circ D_{r^2} \circ \mathcal{I}_{0,1} \circ T_{-a}$.[3]

We can use these isometries to prove that, for any point z_1 on hyperbolic line $l_1\mathcal{H}$, and z_2 on l_2, there is an isometry of $\mathcal{H}$ that can be used to take l_1 to l_2 and z_1 to z_2. This result allows us to prove that similar triangles are congruent in hyperbolic geometry, i.e. if two triangles have the same interior angles in hyperbolic geometry, there is an isometry mapping one to the other (and vice versa).

3. Hyperbolics in Nature

Intuitively, we understand that spherical geometry arises in natural settings where surface area must be minimized—water droplets and bubbles are obvious examples. Knowing this, it might not be surprising to find that hyperbolic geometry arises in settings where surface area must be maximized; after all, the pseudosphere has a Gaussian curvature of -1 where the unit sphere has $K = 1$. A common setting where area maximization occurs is whenever an organism wants to maximize its exchange with a medium, for instance a leaf maximizing its ability to exchange oxygen for carbon dioxide with air. As we alluded to in the half-plane model discussion, these crinkled hyperbolic structures can only be extended so far.[4] However, we can still observe the tell-tale wrinkled structure expected of hyperbolic geometry. Below are two natural examples of hyperbolic geometry arising from this "maximum exchange" goal.

[3]As it turns out, this does not just map half-lines to half-lines and semicircles to semicircles. It maps semicircles centered at a and the geodesic given by $v = a$ to half-lines, and all other geodesics to semicircles.

[4]Specifically, Hilbert proved that surfaces with constant negative Gaussian curvature cannot be "geodesically complete," i.e., they cannot have infinitely extensible geodesics.

FIGURE 2. A holly leaf.[4] Trees photosynthesize, so they need to have high leaf surface are for gas exchange.

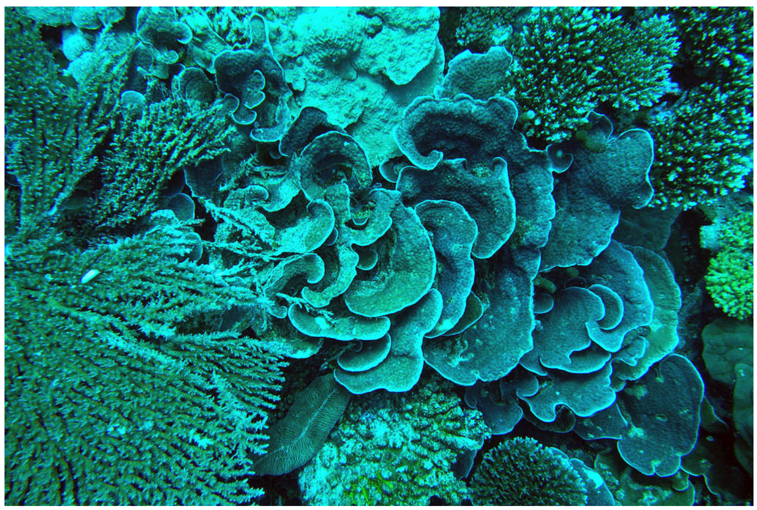

FIGURE 3. A lettuce coral (*Agaricia agaricites*) exhibiting hyperbolic-shaped "leafs." Corals are large colonies of individual organisms, which rely on exchange with water for nutrients and the ions required to build their limestone exoskeletons.[2] Thus, colony structure arises to maximize surface area, thereby maximizing contact with water.

References

[1] Christersson, M. Non-Euclidean geometry. Malin Christersson's Math Site. Retrieved May 17, 2022, from https://www.malinc.se/noneuclidean/en/

[2] National Geographic Society. 2019, August 28. Coral. National Geographic Society. Retrieved May 17, 2022, from https://www.nationalgeographic.org/encyclopedia/coral/

[3] Andrew Pressley, *Elementary Differential Geometry: Second Edition*, 2012, Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84882-891-9_11

[4] Woodlands.co.uk. Holly. Retrieved May 17, 2022, from https://www.woodlands.co.uk/blog/tree-identification/holly/

4. Physical Model

I created a physical model of the hyperbolic plane using equilateral triangles cut out of paper.[5] The first thing to note is that this is a very approximate model; in reality, the hyperbolic plane embedded in 3D would be smooth (of course, it cannot be extended infinitely in Euclidean space). One could improve the precision of this by using a better tiling scheme (I believe one can tile hexagons and heptagons as well), or by making the plane using crochet. I have marked some geodesics in dark lines, and have noted the angles at a few of their intersections. The geodesics do not all intersect, but this is due to the fact that I did not extend the surface far enough (otherwise, there would be parallel lines, which we know to be impossible). The marked angles and geodesic sections enclose a four-sided polygon. The internal angle of this polygon is only 300 degrees, which is consistent with the fact that hyperbolic polygons have a total internal angle of less than 360 degrees. One thing that this model does not do well is that there are straight components of the geodesics across the triangle faces. In reality, these geodesics would not go straight across the face, suddenly bending at intersections. This likely impacts the interior angles of the polygon as well. Lastly, note that every vertex on the plane can be viewed locally as a saddle (just considering the 7 neighboring hexagons). This reflects the fact that the idealized version of this plane has a constant negative Gaussian curvature, resulting in every point being a saddle point (see the discussion in 8.2).

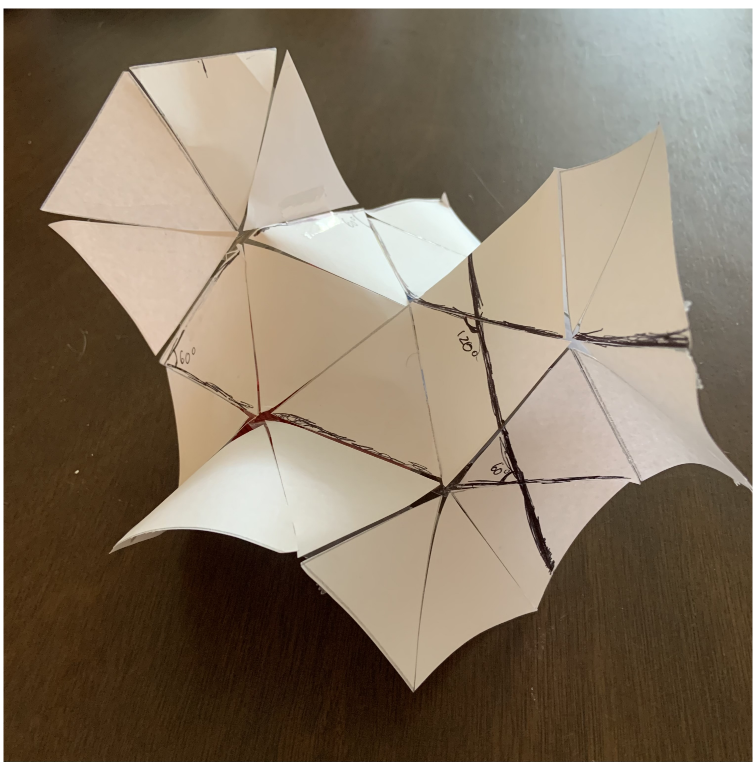

Figure 4. The paper triangle hyperbolic plane model. Note the geodesics written in dark lines. Also note that every vertex looks like a saddle

[5]Sorry Professor, I had said that I was going to make the string model of a hyperboloid of one sheet. I tried to 3D print the parts but it didn't work, so I ended up scrapping it and going with this.

FIGURE 5. The polygon's boundary highlighted. The sum of the interior angles is $60 + 60 + 60 + 120 = 300$ degrees, which is less than the sum of internal angles for a Euclidean polygon.