ECH capacities and fractals of infinite staircases of 4D symplectic embeddings

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based on work with Nicole Magill and Dusa McDuff (arXiv: 2203.06453)
and work in progress with Nicole Magill and Ana Rita Pires

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Convexity in Contact and Symplectic Topology
Institut Henri Poincaré, Paris

We completely describe those Hirzebruch surfaces $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ whose ellipsoid embedding function has an infinite staircase.

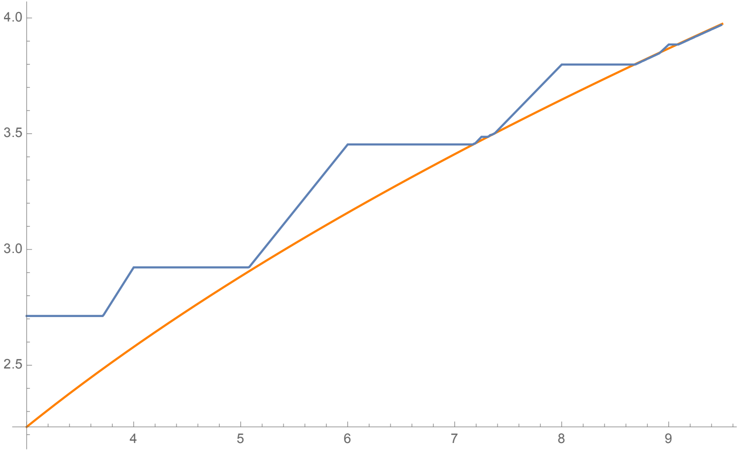

Figure: A new infinite staircase, with many corners between 7 and 8.

Our description is a guide for the case of further toric blowups.

Ellipsoid embedding functions in 4D

The **ellipsoid embedding function** of (X^4, ω) is

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- the Gromov width $c_{\mathrm{Gr}}(X, \omega)$ is $1/c_{(X,\omega)}(1)$
- the structure of $c_{(X,\omega)}$ is delicate and highly variable

Immediate properties of $c_{(X,\omega)}$

$$c_{(X,\omega)}(z) := \inf\left\{ c > 0 \;\middle|\; (E(1,z),\omega_0) \overset{s}{\hookrightarrow} (X, c\omega) \right\}.$$

- $c_{(X,\omega)}$ is piecewise linear or smooth.
- $c_{(X,\omega)}$ is nondecreasing and sublinear.
- volume lower bound:

$$\mathrm{vol}(E(1,z)) \leq \mathrm{vol}(X, c\omega) \Rightarrow c_{(X,\omega)}(z) \geq \sqrt{\frac{z/2}{\mathrm{vol}(X,\omega)}}.$$

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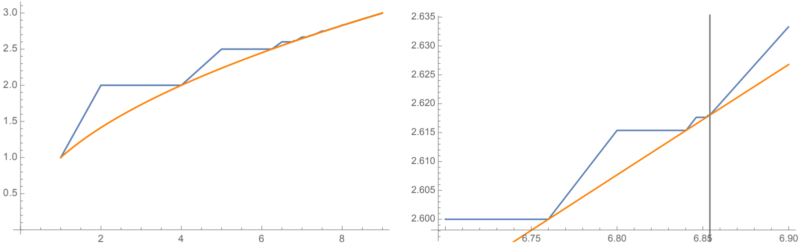
$c_{(X,\omega)}$ has an **infinite staircase** if it is nonsmooth at infinitely many points.

Theorem (McDuff-Schlenk '12)

$c_{B^4(1)}$ has an infinite staircase accumulating from below to (τ^4, τ^2), where $\tau = \frac{1+\sqrt{5}}{2}$.

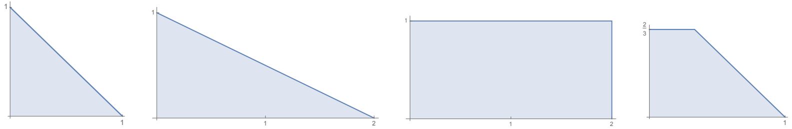


(a) $c_{B^4(1)}$. (b) Zoomed in. $z = \tau^4 = \left(\frac{1+\sqrt{5}}{2}\right)^4$.

Figure: $c_{B^4(1)}$, volume bound $\sqrt{z}$. When $z < \tau^4$, the piecewise linear parts of $c_{B^4(1)}(z)$ are organized into steps above the volume curve consisting of lines through the origin alternating with horizontal lines.

4D convex toric domains

A **(4D) toric domain** X_Ω in $\mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}^2_{\geq 0}$ under the map $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$.



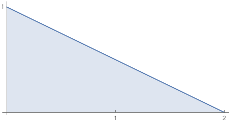
(a) Ball $B(1)$ (b) Ellipsoid $E(1,2)$ (c) Polydisk $P(1,2)$ (d) $X_{\Omega_{\frac{1}{3}}}$

A toric domain is **convex** if Ω is convex in $\mathbb{R}^2$.

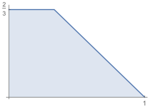
$\mathrm{vol}(X_\Omega, \omega_0) = \mathrm{area}(\Omega)$.

Hirzebruch surfaces and the trapezoid

Let Ω_b be the trapezoid with corner $(b, 1-b)$.



(e) Ellipsoid $E(1,2)$



(f) $X_{\Omega_{\frac{1}{3}}}$

Ω_b is the Delzant polytope of the Hirzebruch surface $\mathbb{C}P^2(1)\#\mathbb{C}P^2(b)$.

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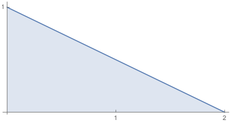

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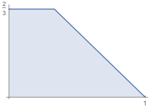

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Cristofaro-Gardiner–Holm–Mandini–Pires '20:

$$c_{X_{\Omega_b}} = c_{\mathbb{C}P^2(1)\#\mathbb{C}P^2(b)}.$$

Notation:

$$c_b(z) := c_{X_{\Omega_b}}(z) = \inf\left\{c>0 \,\middle|\, E(1,z) \overset{s}{\hookrightarrow} X_{c\Omega_b}\right\}.$$

What was known

Rational vertices: CGHMP found twelve Ω with an infinite staircase, including $\Omega_0, \Omega_{1/3}$. Note $X_{\Omega_0} = B^4(1)$.

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- Bertozzi-Holm-Maw-McDuff-Mwakyoma-Pires-W. '21 and Magill-McDuff '21: four bi-infinite families $X_{\Omega_{b_n^k}}$ with infinite staircases; half of them **descending**

Main theorem

Accumulation points

Theorem (CGHMP '20)

An infinite staircase of c_b must accumulate to $\left(acc(b), \sqrt{\frac{acc(b)}{2vol(X_{\Omega_b})}}\right)$, where $acc(b)$ is the larger solution to

$$z^2 - \left(\frac{(3-b)^2}{1-b^2} - 2\right) z + 1 = 0.$$

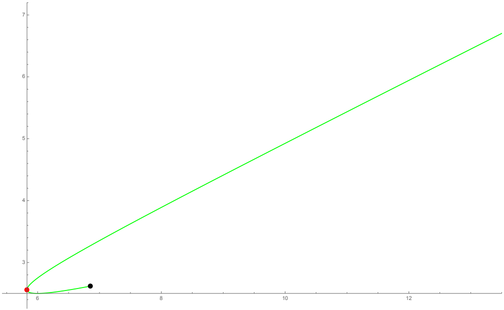
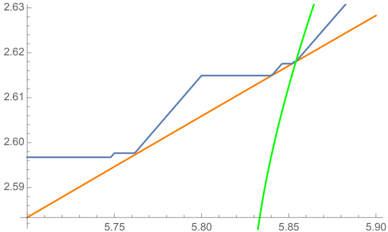
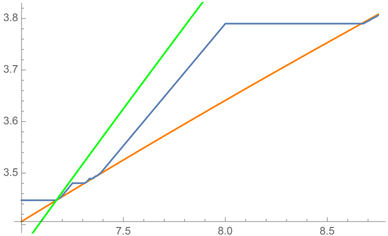


Figure: $\left(\mathrm{acc}(b), \sqrt{\frac{\mathrm{acc}(b)}{2\mathrm{vol}(X_{\Omega_b})}}\right)$, (τ^4, τ^2), $\left(\mathrm{acc}(1/3), \sqrt{\frac{\mathrm{acc}(1/3)}{2\mathrm{vol}(X_{\Omega_{1/3}})}}\right)$.

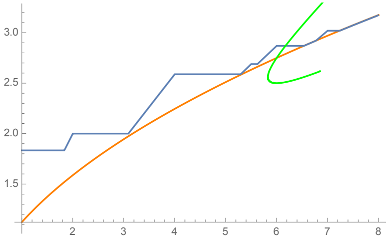
Four possibilities for c_b



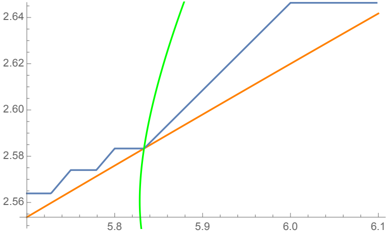
(a) Ascending staircase



(b) Descending staircase



(c) **Blocked**



(d) Unblocked, no staircases

New results: Cantor set of infinite staircases

Theorem (Magill-McDuff-W. '22)

(1) *The set of unblocked* b *with* $acc(b) \in [6, 8]$ *is homeomorphic to the Cantor set.*

(2) *Assume* $acc(b) \in [6, 8]$ *and* b *is not blocked.*

- *If* b *is an endpoint of a blocked interval,* c_b *has either an ascending or a descending infinite staircase.*
- *Otherwise,* c_b *has both.*

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(2) *Assume $acc(b) \in [6,8]$ and b is not blocked.*
- *If b is an endpoint of a blocked interval, c_b has either an ascending or a descending infinite staircase.*
- *Otherwise, c_b has both.*

(3) *All c_b are equivalent to one with $acc(b) \in [6,8]$, except for a countable set with c_b like (d) with no*[a] *infinite staircases.*

(4) *Weak CGHMP conjecture holds: if c_b has an infinite staircase then $acc(b)$ is irrational.*

[a] ascending ruled out in MMW; descending in Magill-Pires-Weiler i.p.

acc(b) $\in [6, 8]$, visualized

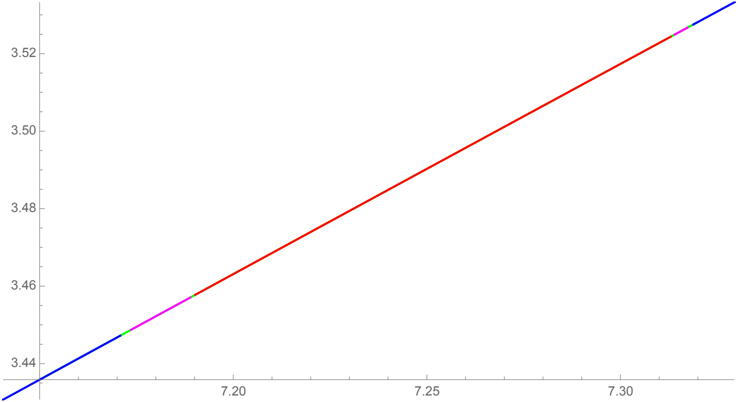

Figure: The accumulation point curve in green. The blocked interval corresponding to $(1/3, 2/3)$ is in red, while those corresponding to $(1/9, 2/9)$ and $(7/9, 8/9)$ are in pink.

Main theorem

Staircase steps are ECH capacity ratios

A metric space's **systole** is the length of its shortest noncontractible curve.

Embedded contact homology provides a $\mathbb{Z}_{\geq 0}$-family of "systoles" called **ECH capacities**: integer linear combinations of actions of Reeb orbits representing classes in the ECH of $\partial X_\Omega = S^3$.

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If Ω is a polygon the c_k are still defined and combinatorial.

Theorem (McDuff '09, Hutchings '11, Cristofaro-Gardiner '19)

If X_Ω is a convex toric domain,

$$\mathrm{int}(E(1,z)) \overset{s}{\hookrightarrow} \mathrm{int}(X_\Omega) \Leftrightarrow c_k(E(1,z)) \leq c_k(X_\Omega)\ \forall k.$$

Staircase steps are ECH capacity ratios

$$c_{X_\Omega}(z) = \sup_k \left\{ \frac{c_k(E(1,z))}{c_k(X_\Omega)} \right\}$$

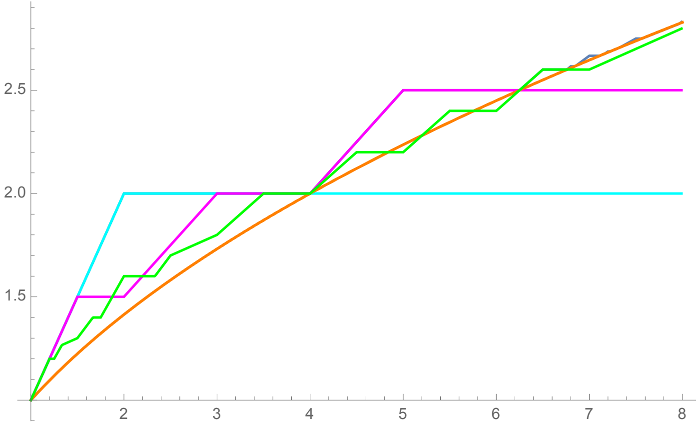

Figure: $\sqrt{z}$, $c_{B^4(1)}(z)$, $\frac{c_2(E(1,z))}{c_2(B^4(1))}$, $\frac{c_5(E(1,z))}{c_5(B^4(1))}$, $\frac{c_{20}(E(1,z))}{c_{20}(B^4(1))}$.

We need many capacities

The capacities of the McDuff-Schlenk Fibonacci stairs grow fast:

$$2, \quad 5, \quad 20, \quad 104, \quad 629, \quad 4,094, \quad 27,494, \quad 186,965, \ \ldots$$

Luckily, if Ω is a quadrilateral we can quickly compute $c_k(X_\Omega)$ for $k \leq 25,000$ (my undergraduate mentee can do better).

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- descending staircases (BHMMMP-W.)
- today's Cantor set (MM-W.)

Exceptional classes

McDuff and McDuff-Schlenk used **exceptional classes**: $E \in H_2(\mathbb{C}P^2 \# M\overline{\mathbb{C}P}^2)$ containing a symplectically embedded sphere.

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$$E(1,z) \overset{s}{\hookrightarrow} X_{c\Omega_b} \Leftrightarrow B^4(cb) \sqcup \bigsqcup_i B^4(w_i) \overset{s}{\hookrightarrow} B^4(c)$$

$$\overset{\text{blow up}}{\Leftrightarrow} \exists\, \omega \text{ on } \mathbb{C}P^2 \# M\overline{\mathbb{C}P}^2,\ \omega(E) > 0\ \forall E,\ \omega(L) = c,$$

where $\mathbf{w}(z) = (w_1, w_2, \dots)$ is the **weight expansion**[1] of z and $M = \ell(\mathbf{w}(z)) + 1$.

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$\omega(E) > 0,\ \omega(L) = c$ gives an obstruction $\mu_{E,b}(z)$, and:

$$c_b(z) = \sup\left\{ \mu_{E,b}(z), \sqrt{\frac{z/2}{\mathrm{vol}(X_{\Omega_b})}} \right\}.$$

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ECH capacities and exceptional classes

If E_0 is the exceptional sphere in the Hirzebruch surface, write $E \in H_2(\mathbb{C}P^2 \# M\overline{\mathbb{C}P}^2)$ as

$$E = (d, m; \mathbf{m}) \leftrightarrow E = dL - mE_0 - \sum_{i=1}^{M-1} m_i E_i,$$

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Monotonicity (Hutchings '11)

$$E(1,z) \overset{s}{\hookrightarrow} X_{c\Omega_b} \Rightarrow c_k(E(1,z)) \leq c_k(X_{c\Omega_b})$$

comes from a count of J-holomorphic curves in the completion of $X_{c\Omega_b} \setminus \varphi(E(1,z))$. Do these curves collapse to exceptional classes?

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In some cases (Cristofaro-Gardiner–Hind '18, C-G–H–McDuff '18), provably yes.

ECH capacities and exceptional classes

$$E = (d, m; \mathbf{m}) \leftrightarrow E = dL - mE_0 - \sum_{i=1}^{M-1} m_i E_i,$$

For us,

$$\frac{c_k(E(1, z))}{c_k(X_{\Omega_b})} = \mu_{E,b}(z)$$

on an interval of z when:

- both are greater than $\sqrt{\frac{z/2}{\mathrm{vol}(X_{\Omega_b})}}$ and equal $c_b(z)$ near $z = p/q$
- $k = \frac{(p+1)(q+1)}{2} - 1$
- $2k = d(d+3) - m(m+1)$
- $\mathbf{m} = \mathbf{w}(p/q)$.

We may thus turn capacities into exceptional classes.

Description of the fractal and proofs

Main Theorem #1

Theorem (Magill-McDuff-Weiler '22)

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Proof.

If c_b is blocked, i.e.

$$c_b(z) > \sqrt{\frac{\mathrm{acc}(b)/2}{\mathrm{vol}(X_{\Omega_b})}},$$

that means there are some k, E so that

$$\frac{c_k(E(1, \mathrm{acc}(b)))}{c_k(X_{\Omega_b})} = \mu_{E,b}(\mathrm{acc}(b)) > \sqrt{\frac{\mathrm{acc}(b)/2}{\mathrm{vol}(X_{\Omega_b})}},$$

an open condition on b. Thus bs are blocked in (disjoint) intervals. □

Continued fractions crash course

$$[a, b, c, d] = a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}$$

$$[a, b, \{c, d\}^2] = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{c + \frac{1}{d}}}}}$$

$$[a, b, \{c, d\}^\infty] = [a, b, c, d, c, d, c, d, \ldots]$$

For example:

$$[7, 4] = 7 + \frac{1}{4} = 7.25,$$

and

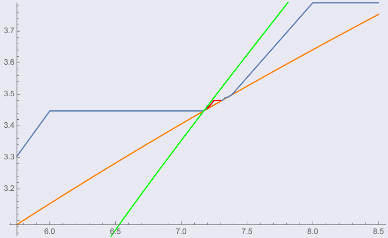
$$[6, \{1, 5\}^\infty] = \tau^4.$$

The fundamental staircases for the Cantor set

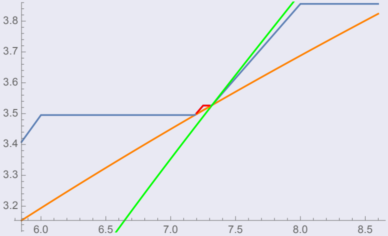
Proof.

BHMMMP-W. found:

- descending infinite staircase, steps at $[8], [7,4], [7,5,2], \ldots$ limiting to the top of the interval blocked by $E = (3, 2; \mathbf{w}(6))$
- ascending infinite staircase, steps at $[6], [7,4], [7,3,6], \ldots$ limiting to the bottom of the interval blocked by $E = (4, 3; \mathbf{w}(8))$.



(a) Descending staircase, $b \approx 0.6297$



(b) Ascending staircase, $b \approx 0.6417$

Proof of Main Theorem #1 and #2.

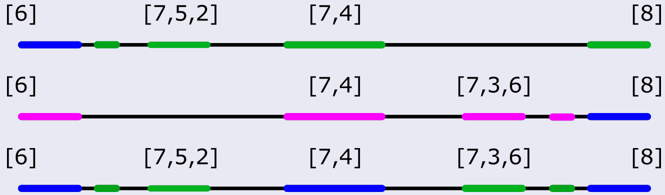


Figure: Horizontal direction: z variable. Colored intervals: acc of the blocked intervals for the step at the label CF.

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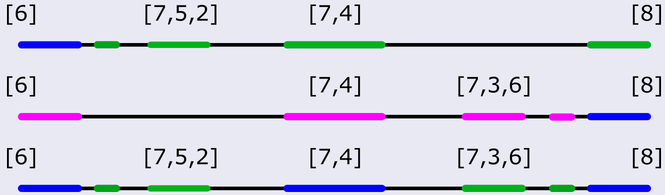

Figure: Horizontal direction: z variable. Colored intervals: acc of the blocked intervals for the step at the label CF.

MM-W. found infinite staircases:

- ascending to the bottom of the interval blocked by $E = (14, 9; \mathbf{w}([7, 4]))$, steps $[6], [7, 5, 2], [7, 5, 3, 1, 6], \ldots$
- descending to the top of the interval blocked by $E = (14, 9; \mathbf{w}([7, 4]))$, steps $[8], [7, 3, 6], [7, 3, 5, 7, 2], \ldots$

Proof of Main Theorem #1 and #2.

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Repeat with the triples $[6], [7, 5, 2], [7, 4]$ and $[7, 4], [7, 3, 6], [8]$ playing the roles of $[6], [7, 4], [8]$.

Repeating forever produces a Cantor set.

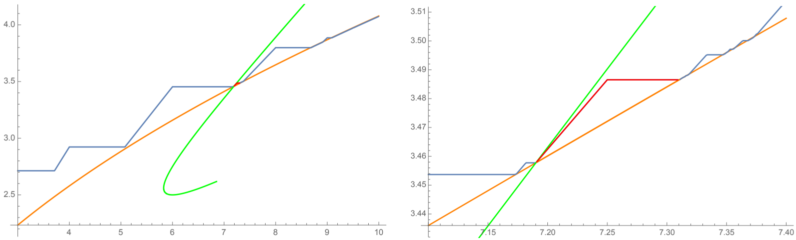
First slide again

The new ascending infinite staircase for the lower endpoint of the interval blocked by $(14, 9; \mathbf{w}([7, 4]))$, with corners

$$[6], [7, 5, 2], [7, 5, 3, 1, 6], [7, 5, 3, 1, 7, 5, 2], \ldots$$

and

$$k = 6, \quad 479, \quad 73,072, \quad 12,113,009, \ \ldots$$



(a) First slide, with accumulation point curve

(b) Zoomed in

c_b with two staircases

Theorem (Magill-McDuff-Weiler '22)

(2) *Assume* $acc(b) \in [6, 8]$ *and b is not blocked.*

- *If b is an endpoint of a blocked interval,* c_b *has either an ascending or descending infinite staircase.*
- *Otherwise,* c_b *has both an ascending and a descending infinite staircase.*

Proof.

We've proved the first bullet.

For the second: if $\mathrm{acc}(b) \in [6, 8]$ is not the endpoint of a blocked interval, infinitely many of the steps obtained via the fractal procedure will still be visible on either side of $\mathrm{acc}(b)$. □

Weak C-G–H–M–P conjecture: Main theorem #4

C-G–H–M–P conjectured: if $b \in \mathbb{Q}, b \neq 0, 1/3$, then c_b does not have an infinite staircase.

We prove: if $\mathrm{acc}(b) \in [6, 8]$ with finite continued fraction, then b is blocked. (Finite continued fractions $\leftrightarrow$ rationals.)

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Future goal: if $\mathrm{acc}(b) \in [6, 8]$ is quadratic irrational (periodic CF), then b is blocked.

Unblocked special rational b: Main theorem #3

Magill-McDuff '21: every $b \in [0,1)$ besides $b_i = \mathrm{acc}^{-1}(y_i/y_{i-1})$

$$y_1 = 1, y_2 = 6, y_i = 6y_{i-1} - y_{i-2}; \quad b_2 = 1/5, b_3 = 11/31, b_4 = 59/179, \ldots$$

has c_b equivalent to some $c_{b'}$ with $\mathrm{acc}(b') \in [6,8]$.

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Magill-Pires-W.: The c_{b_i} do not have descending infinite staircases.

$c_{\frac{11}{31}}$

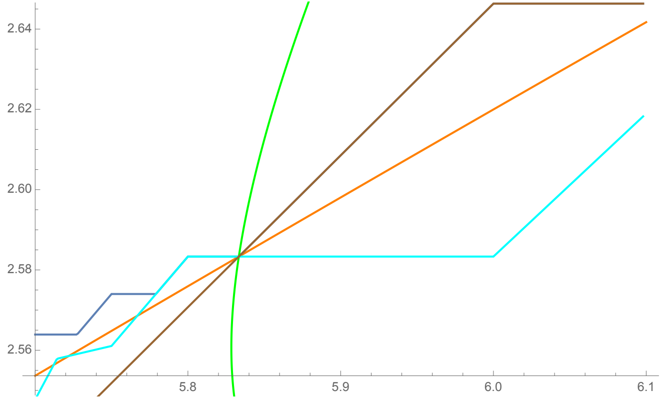

Figure: $c_{\frac{11}{31}}$ near $\mathrm{acc}(11/31) = 35/6 = y_3/y_2$. Volume obstruction, accumulation point curve. They all intersect at $35/6$.

$\frac{c_{89}(E(1,z))}{c_{89}(X_{\Omega_{11/31}})} = \mu_{(13,5;\mathbf{w}(29/5)),11/31}$.

$\frac{c_{8}(E(1,z))}{c_{8}(X_{\Omega_{11/31}})} = \mu_{(3,1;2,1^{\times 5}),11/31}$.

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- A torus action seems crucial for the delicate number-theoretic structure.
- An infinite staircase implies a full filling by the "ECH Weyl law," which is a lot to ask for in general.

The rest of the story

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- Ω has irrational slopes: likely no staircases, see Cristofaro-Gardiner–Salinger for most ellipsoids.
- Many exceptional classes "stabilize," i.e. provide obstructions to embedding into $X_{\Omega_b} \times \mathbb{C}^k$ as in Siegel's talk on Tuesday.

Thank you!