ECH capacities and fractals of infinite staircases of 4D symplectic embeddings

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based on work with Nicole Magill and Dusa McDuff (arXiv: 2203.06453)
and work in progress with Nicole Magill and Ana Rita Pires

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Convexity in Contact and Symplectic Topology
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We completely describe those Hirzebruch surfaces $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ whose ellipsoid embedding function has an infinite staircase.

Figure: A new infinite staircase, with many corners between 7 and 8.

Our description is a guide for the case of further toric blowups.
Ellipsoid embedding functions in 4D

The **ellipsoid embedding function** of \((X^4, \omega)\) is

\[
c_{(X,\omega)}(z) := \inf \left\{ c > 0 \mid (E(1, z), \omega_0) \xrightarrow{s} (X, c\omega) \right\}.
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(McDuff) the maximum fraction of the volume of \((X, \omega)\) that can be filled by \(n \in \mathbb{Z}_{>0}\) equal balls is

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\frac{n}{2c_{(X,\omega)}(n)^2 \text{vol}(X, \omega)}
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\]

- the Gromov width \(c_{Gr}(X, \omega)\) is \(1/c_{(X,\omega)}(1)\)
- the structure of \(c_{(X,\omega)}\) is delicate and highly variable
Immediate properties of $c(X,\omega)$

$$c(X,\omega)(z) := \inf \left\{ c > 0 \mid (E(1, z), \omega_0) \xrightarrow{s} (X, c\omega) \right\}.$$ 

- $c(X,\omega)$ is piecewise linear or smooth.
- $c(X,\omega)$ is nondecreasing and sublinear.
- Volume lower bound:

$$\text{vol}(E(1, z)) \leq \text{vol}(X, c\omega) \Rightarrow c(X,\omega)(z) \geq \sqrt{\frac{z/2}{\text{vol}(X, \omega)}}.$$
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$c(X,\omega)$ has an infinite staircase if it is nonsmooth at infinitely many points.
Theorem (McDuff-Schlenk ’12)

\( c_{B^4(1)} \) has an infinite staircase accumulating from below to \((\tau^4, \tau^2)\), where \( \tau = \frac{1+\sqrt{5}}{2} \).

Figure: \( c_{B^4(1)} \), volume bound \( \sqrt{z} \). When \( z < \tau^4 \), the piecewise linear parts of \( c_{B^4(1)}(z) \) are organized into steps above the volume curve consisting of lines through the origin alternating with horizontal lines.
A (4D) toric domain $X_\Omega$ in $\mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}^2_{\geq 0}$ under the map $(z_1, z_2) \mapsto (\pi |z_1|^2, \pi |z_2|^2)$.

(a) Ball $B(1)$  (b) Ellipsoid $E(1, 2)$  (c) Polydisk $P(1, 2)$  (d) $X_{\Omega^{1/3}}$

A toric domain is **convex** if $\Omega$ is convex in $\mathbb{R}^2$.

$\text{vol}(X_\Omega, \omega_0) = \text{area}(\Omega)$. 
Hirzebruch surfaces and the trapezoid

Let $\Omega_b$ be the trapezoid with corner $(b, 1-b)$.

$\Omega_b$ is the Delzant polytope of the Hirzebruch surface $\mathbb{C}P^2(1) \# \mathbb{C}P^2(b)$.
Hirzebruch surfaces and the trapezoid

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Cristofaro-Gardiner–Holm–Mandini–Pires ’20:

$$c_{\Omega_b} = c_{\mathbb{C}P^2(1) \# \mathbb{C}P^2(b)}.$$

Notation:

$$c_b(z) := c_{\Omega_b}(z) = \inf \left\{ c > 0 \mid E(1, z) \xrightarrow{s} X_{c\Omega_b} \right\}.$$
What was known

Rational vertices: CGHMP found twelve $\Omega$ with an infinite staircase, including $\Omega_0, \Omega_{1/3}$. Note $X_{\Omega_0} = B^4(1)$. 
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**Rational slopes, irrational vertices:**

- **Usher ’19:** bi-infinite family $P(1, b_n^k)$ with infinite staircases
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**Rational slopes, irrational vertices:**
- Usher ’19: bi-infinite family $P(1, b^k_n)$ with infinite staircases
- Bertozzi-Holm-Maw-McDuff-Mwakyoma-Pires-W. ’21 and Magill-McDuff ’21: four bi-infinite families $X_{\Omega_{b^k_n}}$ with infinite staircases; half of them **descending**
Main theorem
Accumulation points

**Theorem (CGHMP ’20)**

An infinite staircase of $c_b$ must accumulate to \( (\text{acc}(b), \sqrt{\frac{\text{acc}(b)}{2\text{vol}(X_{\Omega_b})}}) \), where \( \text{acc}(b) \) is the larger solution to

\[
z^2 - \left( \frac{(3 - b)^2}{1 - b^2} - 2 \right) z + 1 = 0.
\]

Figure: \( (\text{acc}(b), \sqrt{\frac{\text{acc}(b)}{2\text{vol}(X_{\Omega_b})}}), (\tau^4, \tau^2), \left( \text{acc}(1/3), \sqrt{\frac{\text{acc}(1/3)}{2\text{vol}(X_{\Omega_{1/3}})}} \right) \).
Four possibilities for $c_b$

(a) Ascending staircase

(b) Descending staircase

(c) Blocked

(d) Unblocked, no staircases
New results: Cantor set of infinite staircases

**Theorem (Magill-McDuff-W. ’22)**

1. The set of unblocked $b$ with $\text{acc}(b) \in [6, 8]$ is homeomorphic to the Cantor set.
2. Assume $\text{acc}(b) \in [6, 8]$ and $b$ is not blocked.
   - If $b$ is an endpoint of a blocked interval, $c_b$ has either an ascending or a descending infinite staircase.
   - Otherwise, $c_b$ has both.
3. All $c_b$ are equivalent to one with $\text{acc}(b) \in [6, 8]$, except for a countable set with $c_b$ like (d) with no infinite staircases.
4. Weak CGHMP conjecture holds: if $c_b$ has an infinite staircase then $\text{acc}(b)$ is irrational.
   - Ascending ruled out in MMW; descending in Magill-Pires-Weiler i.p.
New results: Cantor set of infinite staircases

Theorem (Magill-McDuff-W. ’22)

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\textsuperscript{a}ascending ruled out in MMW; descending in Magill-Pires-Weiler i.p.
acc(b) ∈ [6, 8], visualized

Figure: The accumulation point curve in green. The blocked interval corresponding to (1/3, 2/3) is in red, while those corresponding to (1/9, 2/9) and (7/9, 8/9) are in pink.
Main theorem
Staircase steps are ECH capacity ratios

A metric space’s **systole** is the length of its shortest noncontractible curve.

Embedded contact homology provides a \( \mathbb{Z}_{\geq 0} \)-family of “systoles” called **ECH capacities**: integer linear combinations of actions of Reeb orbits representing classes in the ECH of \( \partial X_\Omega = S^3 \).

\[
0 = c_0(X_\Omega) < c_1(X_\Omega) \leq c_2(X_\Omega) \leq \cdots \leq \infty.
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A metric space’s **systole** is the length of its shortest noncontractible curve.

Embedded contact homology provides a $\mathbb{Z}_{\geq 0}$-family of “systoles” called **ECH capacities**: integer linear combinations of actions of Reeb orbits representing classes in the ECH of $\partial X_\Omega = S^3$.

$$0 = c_0(X_\Omega) < c_1(X_\Omega) \leq c_2(X_\Omega) \leq \cdots \leq \infty.$$ If $\Omega$ is a polygon the $c_k$ are still defined and combinatorial.

**Theorem (McDuff ’09, Hutchings ’11, Cristofaro-Gardiner ’19)**  

*If $X_\Omega$ is a convex toric domain,*

$$\text{int}(E(1, z)) \overset{s}{\leftrightarrow} \text{int}(X_\Omega) \iff c_k(E(1, z)) \leq c_k(X_\Omega) \forall k.$$
Staircase steps are ECH capacity ratios

\[ c_{X\Omega}(z) = \sup_k \left\{ \frac{c_k(E(1,z))}{c_k(X_{\Omega})} \right\} \]

Figure: \( \sqrt{z}, \ c_{B^4(1)}(z), \ \frac{c_2(E(1,z))}{c_2(B^4(1))}, \ \frac{c_5(E(1,z))}{c_5(B^4(1))}, \ \frac{c_{20}(E(1,z))}{c_{20}(B^4(1))} \).
We need many capacities

The capacities of the McDuff-Schlenk Fibonacci stairs grow fast:

2, 5, 20, 104, 629, 4,094, 27,494, 186,965, ... 

Luckily, if $\Omega$ is a quadrilateral we can quickly compute $c_k(X_\Omega)$ for $k \leq 25,000$ (my undergraduate mentee can do better).
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Large $k$ are helpful for finding:

- the right analogue of Usher’s polydisk symmetries providing the $k$ direction (BHMMMP-W., MM)
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- descending staircases (BHMMMP-W.)
- today’s Cantor set (MM-W.)
Exceptional classes

McDuff and McDuff-Schlenk used exceptional classes: 
\[ E \in H_2(\mathbb{C}P^2 \# M\overline{\mathbb{C}P}^2) \] containing a symplectically embedded sphere.

\[^1 w(z) = w(z, 1), \text{ if } a > b \text{ then } w(a, b) = (b) \cup w(a - b, b)\]
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$E(1, z) \xleftarrow{s} X_{c\Omega_b} \iff B^4(cb) \sqcup \bigsqcup_i B^4(w_i) \xleftarrow{s} B^4(c)$

$\text{blow up} \iff \exists \omega \text{ on } \mathbb{C}P^2 \# M\overline{\mathbb{C}P}^2, \omega(E) > 0 \forall E, \omega(L) = c,$

where $w(z) = (w_1, w_2, \ldots)$ is the weight expansion\(^1\) of $z$ and $M = \ell(w(z)) + 1.$

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where \( w(z) = (w_1, w_2, \ldots) \) is the weight expansion\(^1\) of \( z \) and 
\[ M = \ell(w(z)) + 1. \]

\( \omega(E) > 0, \omega(L) = c \) gives an obstruction \( \mu_{E,b}(z) \), and:

\[ c_b(z) = \sup \left\{ \mu_{E,b}(z), \sqrt{\frac{z/2}{\text{vol}(X_{\Omega_b})}} \right\}. \]

\(^1\)\( w(z) = w(z, 1) \), if \( a > b \) then \( w(a, b) = (b) \cup w(a - b, b) \)
If $E_0$ is the exceptional sphere in the Hirzebruch surface, write $E \in H_2(\mathbb{C}P^2 \# M\mathbb{C}P^2)$ as

\[ E = (d, m; \mathbf{m}) \leftrightarrow E = dL - mE_0 - \sum_{i=1}^{M-1} m_i E_i, \]
ECH capacities and exceptional classes

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Monotonicity (Hutchings ’11)

$$E(1, z) \hookrightarrow X_{c\Omega_b} \Rightarrow c_k(E(1, z)) \leq c_k(X_{c\Omega_b})$$

comes from a count of $J$-holomorphic curves in the completion of $X_{c\Omega_b} \setminus \varphi(E(1, z))$. Do these curves collapse to exceptional classes?
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Monotonicity (Hutchings ’11)

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comes from a count of $J$-holomorphic curves in the completion of $X_{c\Omega_b} \setminus \phi(E(1, z))$. Do these curves collapse to exceptional classes? In some cases (Cristofaro-Gardiner–Hind ’18, C-G–H–McDuff ’18), provably yes.
ECH capacities and exceptional classes

\[ E = (d, m; \mathbf{m}) \iff E = dL - mE_0 - \sum_{i=1}^{M-1} m_i E_i, \]

For us,

\[ \frac{c_k(E(1, z))}{c_k(X_{\Omega_b})} = \mu_{E,b}(z) \]

on an interval of \( z \) when:

- both are greater than \( \sqrt{\frac{z/2}{\text{vol}(X_{\Omega_b})}} \) and equal \( c_b(z) \) near \( z = p/q \)
- \( k = \frac{(p+1)(q+1)}{2} - 1 \)
- \( 2k = d(d + 3) - m(m + 1) \)
- \( \mathbf{m} = w(p/q) \).

We may thus turn capacities into exceptional classes.
Description of the fractal and proofs
Main Theorem #1

Theorem (Magill-McDuff-Weiler ’22)

(1) The set of unblocked \( b \) with \( \text{acc}(b) \in [6, 8] \) is homeomorphic to the Cantor set.
Main Theorem #1

### Theorem (Magill-McDuff-Weiler '22)

(1) *The set of unblocked* $b$ *with* $\text{acc}(b) \in [6, 8]$ *is homeomorphic to the Cantor set.*

### Proof.

If $c_b$ is blocked, i.e.

\[
 c_b(z) > \sqrt{\frac{\text{acc}(b)/2}{\text{vol}(X_{\Omega_b})}},
\]

that means there are some $k, E$ so that

\[
 \frac{c_k(E(1, \text{acc}(b)))}{c_k(X_{\Omega_b})} = \mu_{E,b}(\text{acc}(b)) > \sqrt{\frac{\text{acc}(b)/2}{\text{vol}(X_{\Omega_b})}},
\]

an open condition on $b$. Thus $b$s are blocked in (disjoint) intervals.
Continued fractions crash course

\[
[a, b, c, d] = a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}
\]

\[
[a, b, \{c, d\}^2] = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{c + \frac{1}{d}}}}}
\]

\[
[a, b, \{c, d\}^\infty] = [a, b, c, d, c, d, c, d, \ldots]
\]

For example:

\[
[7, 4] = 7 + \frac{1}{4} = 7.25,
\]

and

\[
[6, \{1, 5\}^\infty] = \tau^4.
\]
The fundamental staircases for the Cantor set

**Proof.**

BHMMMP-W. found:

- descending infinite staircase, steps at [8], [7, 4], [7, 5, 2], . . . limiting to the top of the interval blocked by $E = (3, 2; w(6))$
- ascending infinite staircase, steps at [6], [7, 4], [7, 3, 6], . . . limiting to the bottom of the interval blocked by $E = (4, 3; w(8))$.

(a) Descending staircase, $b \approx 0.6297$

(b) Ascending staircase, $b \approx 0.6417$
Proof of Main Theorem #1 and #2.

**Figure:** Horizontal direction: $z$ variable. Colored intervals: acc of the blocked intervals for the step at the label CF.

Repeating forever produces a Cantor set.
Proof of Main Theorem #1 and #2.

MM-W. found infinite staircases:

- ascending to the bottom of the interval blocked by $E = (14, 9; w([7, 4]))$, steps [6], [7, 5, 2], [7, 5, 3, 1, 6], ...
- descending to the top of the interval blocked by $E = (14, 9; w([7, 4]))$, steps [8], [7, 3, 6], [7, 3, 5, 7, 2], ...

Figure: Horizontal direction: $z$ variable. Colored intervals: acc of the blocked intervals for the step at the label CF.
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- ascending to the bottom of the interval blocked by $E = (14, 9; w([7, 4]))$, steps $[6], [7, 5, 2], [7, 5, 3, 1, 6], \ldots$
- descending to the top of the interval blocked by $E = (14, 9; w([7, 4]))$, steps $[8], [7, 3, 6], [7, 3, 5, 7, 2], \ldots$

Repeat with the triples $[6], [7, 5, 2], [7, 4]$ and $[7, 4], [7, 3, 6], [8]$ playing the roles of $[6], [7, 4], [8]$.

Repeating forever produces a Cantor set.
The new ascending infinite staircase for the lower endpoint of the interval blocked by \((14, 9; w([7, 4]))\), with corners

\[[6], [7, 5, 2], [7, 5, 3, 1, 6], [7, 5, 3, 1, 7, 5, 2], \ldots\]

and

\[k = 6, 479, 73,072, 12,113,009, \ldots\]

(a) First slide, with accumulation point curve

(b) Zoomed in

Fractals of infinite staircases
$c_b$ with two staircases

**Theorem (Magill-McDuff-Weiler ’22)**

(2) Assume $\text{acc}(b) \in [6, 8]$ and $b$ is not blocked.

- If $b$ is an endpoint of a blocked interval, $c_b$ has either an ascending or descending infinite staircase.
- Otherwise, $c_b$ has both an ascending and a descending infinite staircase.

**Proof.**

We’ve proved the first bullet.

For the second: if $\text{acc}(b) \in [6, 8]$ is not the endpoint of a blocked interval, infinitely many of the steps obtained via the fractal procedure will still be visible on either side of $\text{acc}(b)$.
Weak C-G–H–M–P conjecture: Main theorem #4

C-G–H–M–P conjectured: if \( b \in \mathbb{Q}, b \neq 0, 1/3 \), then \( c_b \) does not have an infinite staircase.

We prove: if \( \text{acc}(b) \in [6, 8] \) with finite continued fraction, then \( b \) is blocked. (Finite continued fractions \( \leftrightarrow \) rationals.)
Weak C-G–H–M–P conjecture: Main theorem #4

C-G–H–M–P conjectured: if $b \in \mathbb{Q}$, $b \neq 0, 1/3$, then $c_b$ does not have an infinite staircase.

We prove: if $\text{acc}(b) \in [6, 8]$ with finite continued fraction, then $b$ is blocked. (Finite continued fractions $\leftrightarrow$ rationals.)

Future goal: if $\text{acc}(b) \in [6, 8]$ is quadratic irrational (periodic CF), then $b$ is blocked.
Magill-McDuff ’21: every $b \in [0, 1)$ besides $b_i = \text{acc}^{-1}(y_i/y_{i-1})$

$y_1 = 1, y_2 = 6, y_i = 6y_{i-1} - y_{i-2}; \quad b_2 = 1/5, b_3 = 11/31, b_4 = 59/179, \ldots$

has $c_b$ equivalent to some $c_{b'}$ with $\text{acc}(b') \in [6, 8]$.

**Theorem (Magill-McDuff-Weiler ’22)**

The $c_{b_i}$ do not have ascending infinite staircases.
Unblocked special rational $b$: Main theorem #3

Magill-McDuff ’21: every $b \in [0, 1)$ besides $b_i = \text{acc}^{-1}(y_i/y_{i-1})$

$y_1 = 1$, $y_2 = 6$, $y_i = 6y_{i-1} - y_{i-2}$; $b_2 = 1/5$, $b_3 = 11/31$, $b_4 = 59/179$, …

has $c_b$ equivalent to some $c_{b'}$ with $\text{acc}(b') \in [6, 8]$.

**Theorem (Magill-McDuff-Weiler ’22)**

The $c_{b_i}$ do not have ascending infinite staircases.

**Magill-Pires-W.:** The $c_{b_i}$ do not have descending infinite staircases.
Figure: $c_{\frac{11}{31}}$ near $\text{acc}(\frac{11}{31}) = \frac{35}{6} = \frac{y_3}{y_2}$. Volume obstruction, accumulation point curve. They all intersect at $\frac{35}{6}$.

\[
\frac{c_{89}(E(1,z))}{c_{89}(X_{\Omega_{11/31}})} = \mu(13,5;w(29/5)),11/31. \\
\frac{c_8(E(1,z))}{c_8(X_{\Omega_{11/31}})} = \mu(3,1;2,1^{\times 5}),11/31. 
\]
The role of convexity

- When the target is a concave toric domain, embedding constructions (upper bounds) don't work.
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- When the target is a concave toric domain, embedding constructions (upper bounds) don't work.
- When the domain is a convex toric domain, the $c_k$ ratios don't equal $c(X,\omega)$ (Hind-Lisi ’15, see also Hutchings ’15).
The role of convexity

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- When the domain is a convex toric domain, the $c_k$ ratios don’t equal $c_{(\chi,\omega)}$ (Hind-Lisi ’15, see also Hutchings ’15).
- A torus action seems crucial for the delicate number-theoretic structure.
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- When the domain is a convex toric domain, the $c_k$ ratios don’t equal $c(\chi,\omega)$ (Hind-Lisi ’15, see also Hutchings ’15).
- A torus action seems crucial for the delicate number-theoretic structure.
- An infinite staircase implies a full filling by the “ECH Weyl law,” which is a lot to ask for in general.
The rest of the story

- In progress:
  - Prove the full CGHMP conjecture for $\Omega_b$ (rule out rational $b$).
  - Rule out a descending staircase for $\Omega_{1/3}$.
  - 2022 with Holm, Magill, and 4 undergraduates: there's a polydisk analogue of the ascending $[7, 4]$ staircase.
  - Work in progress by Magill suggests that the target $\mathbb{CP}^2 \# \mathbb{CP}^2 (b_1) \# \mathbb{CP}^2 (b_2)$ contains curves of pairs $(b_1, b_2)$ whose classification into infinite staircase/no infinite staircase is based on our description and interpolates between the trapezoid and polydisk.

$\Omega$ has irrational slopes: likely no staircases, see Cristofaro-Gardiner–Salinger for most ellipsoids.

Many exceptional classes "stabilize," i.e. provide obstructions to embedding into $X_\Omega \times \mathbb{C}^k$ as in Siegel's talk on Tuesday.
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\[ \text{Morgan Weiler Cornell University} \hspace{1cm} \text{Fractals of infinite staircases} \]
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Thank you!