1. Introduction

My research combines contact and symplectic geometry with low-dimensional topology to study dynamics in two, three, and four dimensions. The invariant I use, embedded contact homology (ECH), is powerful but difficult to compute. There are two filtrations on ECH that induce spectral invariants: the ECH capacities (1.1), derived from the symplectic action, are used to obstruct symplectic embeddings, while knot-filtered ECH (2), derived from linking number, is used to study surface dynamics and the knotting of Reeb orbits. Computing either requires understanding the ECH chain complex, so a priori cannot pass through either of the isomorphic theories of Seiberg-Witten or Heegaard Floer homologies.

My research leverages low-dimensional topology and toric geometry to construct, compute, and apply ECH spectral invariants to geometric problems. I follow two major directions. Firstly, I work to extend the foundations of knot-filtered ECH (§2.2), along the way providing computations of this relatively new invariant (§2.1). Secondly, I am interested in codifying the relationship between ECH cobordism maps and other symplectic embedding obstructions while computing new embedding capacity functions (§3.3) and solving a case of a conjecture of Cristofaro-Gardiner–Holm–Mandini–Pires classifying symplectic embeddings \[ \text{[10]} \] (§3.2 §3.3).

1.1. Central definitions. We use \( Y \) to denote a smooth, closed, oriented three-manifold. A contact form is a one-form \( \lambda \) on \( Y \) with \( \lambda \wedge d\lambda > 0 \). If \( Y \subset \mathbb{C}^2 \) then often \( \lambda \vert_Y \) is a contact form, where \( \lambda_0 = \frac{1}{2} \sum_{i=1}^2 x_idy_i - y_idx_i \). Contact forms induce Reeb vector fields: smooth nonzero vector fields on \( Y \) determined by \( d\lambda(R, \cdot) = 0 \) and \( \lambda(R) = 1 \). Closed integral curves of the Reeb vector field, i.e. \( \gamma : \mathbb{R}/\mathbb{R} \to Y \) with \( \gamma'(t) = R_{\gamma(t)} \), modulo reparameterization, are called Reeb orbits, which are embedded when they are injective. The (symplectic) action of a Reeb orbit \( \gamma \) is \( \int_\gamma \lambda \).

My main tool is ECH, a Floer homology theory developed by Hutchings [18] and generated by \( \alpha \). When \( \alpha \) and \( \beta \) are ECH generators, the coefficient of \( \beta \) in \( \partial \alpha \) is a count of \( J \)-holomorphic curves asymptotic to cylinders over the orbits in \( \alpha \) and \( \beta \) as the \( \mathbb{R} \)-coordinate goes to \( \pm \infty \). At the homology level, ECH depends only on \( (Y, \ker \lambda) \) and not \( J \) [29]; thus we denote the ECH of \( (Y, \lambda) \) by \( \text{ECH}_*(Y, \ker \lambda) \). ECH was used to prove the three-dimensional Weinstein conjecture [28], characterize symplectic embeddings [24], and satisfies a Weyl law [11]. There are two key spectral invariants in ECH, ECH capacities and knot-filtered ECH, derived from filtrations, i.e. functions on the ECH chain complex which decrease with respect to the differential. To compute either spectral invariant, we need to compute the entire ECH chain complex directly from \( \lambda \).

A symplectic form on a smooth 2n-manifold \( X \) is a smooth two-form \( \omega \) with \( d\omega = 0, \omega^n \neq 0 \). If \( \omega = d\lambda \) and \( \lambda \vert_{\partial X} \) is a contact form, the ECH capacities of \( (X, \omega) \) are

\[
0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty,
\]

where \( c_k(X, \omega) \) measures the least symplectic action necessary to realize certain classes in \( \text{ECH}_{2k}(\partial X, \lambda) \) by homologically essential cycles of Reeb orbits for a contact form primitive of \( \omega \) [17].

ECH capacities obstruct symplectic embeddings from \( (X, \omega) \) to \( (X', \omega') \), which are smooth embeddings \( \phi : X \to X' \) with \( \phi^*\omega' = \omega \). We use the notation \( X \xrightarrow{k} X' \) when \( \omega, \omega' \) are understood. The isomorphism between ECH and Seiberg-Witten Floer homology [28] [20] implies

\[
X \xrightarrow{k} X' \Rightarrow c_k(X, \omega) \leq c_k(X', \omega') \forall k.
\]
2. knot-filtered ECH

In my 2019 thesis, I investigated knot-filtered ECH, a spectral invariant which measures the linking of ECH generators with a fixed Reeb orbit. It was introduced by Hutchings in [19] for $Y$ with $H_1(Y) = 0$. We denote by $ECH^\ell(Y, \ker \lambda, B, \theta)$, where $B$ is a Reeb orbit, $\ell \in \mathbb{R}$, and $\theta$ is a parameter depending on $d\lambda$ near $B$, the homology of the subcomplex generated by orbits of linking number at most $\ell$ with $B$. knot-filtered ECH likely exists for general $Y$, and in my thesis I defined it when $b_1(Y) = 0$ and proved invariance of $\lambda$ and $J$:

**Theorem 2.1** (W ’19 [31]). If $b_1(Y) = 0$, $ECH^\ell(Y, \ker \lambda, B, \theta)$ is defined and independent of $\lambda, J$.

I applied Thm. 2.1 to lens spaces $Y = L(p, p - 1)$ with $B$ either component of the image of the Hopf link under the quotient $L(p, p - 1) = S^3/\mathbb{Z}_p$, computed their knot-filtered ECH, and proved:

**Theorem 2.2** (W ’19 [31]). Such $Y$ have a Reeb orbit $\gamma$ whose action has an upper bound given by its linking numbers with the components of image of the Hopf link and $\int_Y \lambda \wedge d\lambda, p, \theta$.

Bechara Senior–Hryniewicz–Salomão have generalized Thm. 2.2 to generic $(Y, \lambda)$ [2].

In my thesis [31] I used Thm. 2.2 to prove a quantitative smooth version of Franks’ theorem [14] that area-preserving homeomorphisms of the annulus have two or infinitely many periodic orbits:

**Theorem 2.3** (W ’19 [31]). If $\psi$ is a symplectomorphism of the annulus, a rotation near the boundary, and its “Calabi invariant” (an average measure of rotation) is less than its larger boundary rotation number, then $\psi$ has infinitely many periodic orbits.

I proved Thm. 2.3 by embedding the annulus transversally to the Reeb vector field of a contact $L(p, p - 1)$ as in Thm. 2.2 realizing $\psi$ as the return map of the Reeb flow from the annulus to itself. This method was inspired by Hofer’s “global surfaces of section” [16] and the case of the disk in [19]; see also [1] for a proof of the smooth Hamiltonian closing lemma using similar methods.

To date, direct computations of ECH require an effective torus action on $Y$ preserving $\lambda$. In order to analyze more complex $(Y, \lambda)$, it is essential that we understand the ECH chain complex of non-toric $(Y, \lambda)$. My work on the ECH of circle bundles, joint with Jo Nelson, is the first step.

**Theorem 2.4** (NW [27]). Let $Y$ be an $S^1$ bundle over a surface with a contact form $\lambda$ whose Reeb flow is the fiber action. Then $ECH_\ast(Y, \ker \lambda)$ is the exterior algebra of the homology of the surface.

The contact forms studied in Thm. 2.4 exist on all circle bundles [5]. Thm. 2.4 first appeared in a $\mathbb{Z}_2$-graded form in the unpublished thesis of Farris [12]: Nelson and I completed the $J$-holomorphic curve analysis and proved the $\mathbb{Z}_2$-graded version. Our work was used by Ferreira-Ramos to compute the ECH capacities of some disk bundles in [13].

2.1. New computations. I have computed new examples of knot-filtered ECH:

**Example 2.5** (NW [26]). Denote the positive trefoil by $T_{3,2}$. If $\theta \approx 6$, then $ECH^\ell_\ast(S^3, \ker \lambda_0|_{S^3}, T_{3,2}, \theta)$ is nonzero if and only if $\ell > c_6(E(3, 2))$, where $E(3, 2)$ is the “ellipsoid” defined in §3.

Understanding the knot-filtered of other $\lambda$ will require new computational methods. There is a Seifert fibration of $S^3$ with $T_{\pm 3,2}$ as a regular fiber, inducing a contact form with the $S^3$ action as the Reeb flow. By viewing the Reeb flow as an $S^1$-bundle over an orbifold, Nelson and I will generalize Thm. 2.4 to compute the knot-filtered ECH of the left-handed trefoil:

**Project 2.6** (NW [26]). Compute $ECH^\ell_\ast(S^3, \ker \lambda_0|_{S^3}, T_{-3,2}, \theta)$ for $\theta \approx -6$.

Although the Reeb vector fields in Ex. 2.5 and Proj. 2.6 are isomorphic as Seifert fibrations, their contact structures are far from contactomorphic: one is tight and the other overtwisted. Project 2.6 will thus reveal more of the dependence of knot-filtered ECH on the contact structure.
2.2. Foundations and relations to Heegaard Floer theory. In order to study the linking properties of Reeb orbits for general $Y$—in particular, circle bundles on surfaces, whose ECH chain complex we analyzed to prove Thm. 2.4— I am extending the foundations of knot-filtered ECH:

**Project 2.7 (Weiler).** Prove knot-filtered ECH is defined for all $Y$, and independent of $\lambda, J$.

Knot (Heegaard) Floer homology contains two invariants, $\tau$ and $\Upsilon$, whose constructions are reminiscent of that of knot-filtered ECH. In particular, $\tau$ is similar to the smallest value $\ell$ for which $ECH^\ell_0(S^3, \ker \lambda, K, \theta)$ is nonzero, although $\tau$ does not depend on the parameter $\theta$ recording the contact form’s differential $d\lambda$ near $K$. The invariant $\Upsilon$ extends $\tau$ using a $U$-filtration which also appears in ECH. Therefore, I will investigate whether $\tau$ and $\Upsilon$ can be computed from knot-filtered ECH, perhaps by fixing or limiting over the parameter $\theta$.

3. Symplectic embeddings and the ECH spectrum

My research program also investigates symplectic embedding problems, for which ECH invariants have proved very useful (see [7]). Symplectic forms induce volumes: we define $\text{vol}(X, \omega) := \int_X \omega^\wedge n$, and $(X, \omega) \hookrightarrow (X', \omega')$ only if $\text{vol}(X, \omega) \leq \text{vol}(X', \omega')$. Therefore a central theme in symplectic geometry is discerning how it is “rigid,” i.e. similar to complex geometry, versus “flexible,” i.e. similar to volume-preserving geometry. Symplectic embeddings are at the heart of this rigidity-flexibility dichotomy. With his proof of “Gromov nonsqueezing,” Gromov proved that similar to volume-preserving geometry. Symplectic embeddings are at the heart of this rigidity-flexibility dichotomy. With his proof of “Gromov nonsqueezing,” Gromov proved that $B(r) \hookrightarrow Z(R) \Rightarrow r \leq R$, where $B(r)$ is the finite-volume ball (Fig. 1 (a)) and $Z(R)$ is the infinite-volume cylinder $\{z \in \mathbb{C}^2| |z_1|^2 < R\}$ with the symplectic form $\omega_0$ (Def. 3.1) [15]. Nonsqueezing is a rigid phenomenon because embeddings are obstructed by just one local parameter, the area of each manifold’s intersection with the $\mathbb{C} \times \{0\}$ axis. Yet there are also flexible cases, meaning that volume and topology are the only obstruction to embedding: [1], [27]. My work investigates the exact boundary between rigidity and flexibility.

![Figure 1. Several $\Omega$ determining toric domains (Def. 3.1), with $X\Omega$ named. All are convex; ellipsoids are also concave.](image)

**Definition 3.1.** For $\Omega$ a region in the first quadrant of $\mathbb{R}^2$, the **toric domain** $(X\Omega, \omega_0)$ is

$$X\Omega := \{(z_1, z_2) \in \mathbb{C}^2|(|z_1|^2, |z_2|^2) \in \Omega\}, \quad \omega_0 = \sum_{i=1}^2 dx_i \wedge dy_i|_{X\Omega}.$$  

If $\Omega$ is the region under the graph of a concave (resp., convex) function, $X\Omega$ is convex (resp., concave), and $\lambda_0|\partial Y$ is a contact form on $\partial X$ with $d\lambda_0 = \omega_0$. See Fig. 1 for key examples.

3.1. **Past work.** Even the question of whether $E(a, b) \hookrightarrow E(c, d)$ is very difficult to answer [23]. In 2020 work with Bertozzi, Holm, Maw, Mwakyoma, McDuff, and Pires, I analyzed symplectic embeddings into a continuous range of toric domains, discovering delicate and complex behaviors. Because we always have $E(a, 1) \hookrightarrow cX$ if $c >> 0$, we study embeddings of ellipsoids into $X$ via the **ellipsoid embedding function**

$$c_X(a) := \inf \{c|(E(a, 1), \omega_0) \hookrightarrow (X, c\omega_0)\},$$
first investigated by McDuff and Schlenk [25]. They found $c_{B(1)}$ contained an infinite staircase, i.e., $c_{B(1)}$ is smooth or piecewise linear, and contains infinitely many nonsmooth points whose a-coordinates accumulate at the golden ratio to the fourth. Ellipsoid embedding functions always satisfy the lower bound

$$c_X(a) \geq \text{vol}_X(a),$$

where $\text{vol}_X(a)$ is the smallest value of $c$ so that $\text{vol}(X,c_\omega) = \text{vol}(E(1,a),\omega_0)$. The numerics of this staircase describe regions of alternately rigid and flexible embeddings: when (3.1) is a strict inequality, all embeddings $(E(1,a),\omega_0) \overset{s}{\rightarrow} (X,c_\omega)$ are rigid, whereas the embedding $(E(1,a),\omega_0) \overset{s}{\rightarrow} (X,\text{vol}_X(a),\omega_0)$ is flexible, filling all available volume.

Previous researchers have investigated $c_X$ for other convex $X$, discovering some with infinite staircases and some without: [30] [10] [8]. Although there are shared techniques, small adjustments to $\Omega$ can drastically change the problem. With Bertozzi, Holm, Maw, Mwakyoma, McDuff, and Pires, I studied $c_{H_b}$, where $H_b$ is the complex projective plane blown up once with weight $b \in [0,1)$. We have $c_{H_b} = c_{X_b}$ (Fig. 1 (d)) by [10] Thm. 1.3, allowing us to use ECH tools to study $c_{H_b}$.

**Theorem 3.2 (BHMMMPW [3]).** $c_{H_b}$ has an infinite staircase for six infinite families of $b$ values.

We identified the values of $b$ in Thm. 3.2 were with computer programs approximating $c_{H_b}$ via ECH capacities. McDuff proved (1.1) is an equivalence for $X = E(a,b),X' = E(c,d)$ [24], and Cristofaro-Gardiner generalized her work to show (1.1) is an equivalence if $X = X_\Omega$ is convex and $X' = X_{\Omega'}$ is convex [7]. The ECH capacities of convex and concave $X_\Omega$ are combinatorially computable, but using (1.1) to compute $c_{X_\Omega}$ exactly is generally impossible, because it requires understanding an infinite sequence of capacities. The best we can do is approximate $c_{X_\Omega}$, and this is still difficult, because the new infinite staircases are only visible if tens of thousands of ECH capacities have been computed. Previously, researchers had only been able to compute hundreds or thousands of capacities in any reasonable time (10, 30), leaving the staircases of Thm. 3.2 invisible. Our work leveraged direct computations of the ECH of $\partial(X_b)$ to optimize the algorithms for $c_k(X_b)$, illuminating our new infinite staircases, as well as some of those later investigated by Magill-McDuff (see next paragraph) and Magill-McDuff-Weiler (Thm. 3.3). Another new feature of our work in [3] is the discovery of descending infinite staircases: all previously known infinite staircases accumulated from below.

Thm. 3.2 was recently extended to parallel Usher’s results on embeddings into polydisks [30] and include descending staircases (which Usher did not discuss, but which conjecturally also appear in the polydisk case): in [21], Magill–McDuff found a bi-infinite family of values of $b$ for which $c_{H_b}$ has an infinite staircase, extending the original six. In recent work [22] with Magill and McDuff, I have further extended this bi-infinite family, uncovering uncountably many new staircases with no parallel to Usher’s:

**Theorem 3.3 (Magill–McDuff–Weiler [22]).** There is an open dense subset $B \subset [6,8]$ such that:

(i) $[6,8] \setminus B$ is homeomorphic to the Cantor set.

(ii) For each $a \in [6,8] \setminus B$, there is an ascending or descending infinite staircase, or both, with accumulation point $(a,\text{vol}_X(a))$.

(iii) The generalizations of [21] extend (ii) to determine for all $b \in [0,1)$ whether $c_{H_b}$ has an infinite staircase, except perhaps one countable family with descending infinite staircases.

We expect the potential descending staircases in (iii) do not exist: see \S 3.2.

3.2. Cristofaro-Gardiner–Holm–Mandini–Pires conjecture. In [10], Cristofaro-Gardiner–Holm–Mandini–Pires conjecture that up to scale there are only 12 convex $X_\Omega$ where $\Omega$ is a polygon with corners in $\mathbb{Q}^2$ for which $c_{X_{\Omega}}$ has an infinite staircase. They also showed that the a-coordinate
of any infinite staircase $c_{H_b}$ with rational $b$ must be either rational or a quadratic irrational. With Magill and McDuff, I have proved part of this conjecture:

**Theorem 3.4** (Magill–McDuff–Weiler [22]). The only values of $b$ where $c_{H_b}$ has an ascending infinite staircase and an accumulation point with rational $a$-coordinate are $b = 0, \frac{1}{3}$.

In future work we plan to extend Thm. 3.4 to rule out descending infinite staircases (Thm. 3.3 (iii)) as well as quadratic irrational $a$-coordinates of accumulation points, thus proving the conjecture for $X_b$ (and thus $H_b$). We hope our work will provide a blueprint for proving the conjecture more generally (ellipsoids were treated in [6]; our methods should extend to polydisks).

### 3.3. ECH cobordism maps and embedding obstructions

When $\Omega$ is the region under the graph of a concave piecewise linear function with rational edge slopes, $c_{X_\Omega}$ can be computed as the supremum over an infinite family of either ratios of ECH capacities (as described above), or over an infinite family of obstruction functions determined by symplectically embedded spheres in a blowup of the symplectic toric manifold obtained by collapsing a $S^1$-fibration of $\partial(\Omega)$; this method was pioneered by [25] and extended in [10].

The heuristic relationship between the two perspectives is the following. If $\varphi : (X, \omega) \xrightarrow{s} (X', \omega')$, then $X' \setminus \varphi(X)$ is a symplectic cobordism. A natural way to define cobordism maps from the ECH of $\partial X'$ to that of $\varphi(X)$ (equivalently $\partial X$, since ECH is a diffeomorphism invariant) is via counting $J$-holomorphic curves in $(X' \setminus \varphi(X), \omega')$. ECH capacities of $X$ and $X'$ would differ by at least the symplectic areas of these curves, and in nice cases, by exactly these areas. When $X, X'$ are toric domains with rational edge slopes, $\partial X, \partial(X)$ can be collapsed along the $S^1$-fibration by the Reeb vector fields of their natural contact forms. Any $J$-holomorphic curves collapse to symplectically embedded surfaces, and if they are spheres and $X$ is an ellipsoid, the obstruction to $c_{X'}$ from the cobordism map equals that from the image of the collapsed $J$-holomorphic curve. This perspective was initiated in [2].

Experimentally, the above heuristic holds. In all cases checked, for certain intervals of $a$ values the lower bound on $c_{X'}$, derived from the $k^{th}$ ECH capacity equals the lower bound on $c_{X'}$, derived from a symplectically embedded sphere in the correct second homology class to arise from collapsing the boundary of a curve counted by the ECH cobordism map in index $2k$. In [3, Lem. 92] we proved one direction of this correspondence when $X = H_b$.

I plan to explore this potential equivalence further. There are several results in the literature which have only been proved using either ECH capacities or symplectically embedded spheres, while using the other method would extend the result. Most significant is a Cristofaro-Gardiner–Holm–Mandini–Pires result providing the coordinates of the accumulation point of an infinite staircase ([10, Thm. 1.11]), which is crucial to all the work described in [3.1]. It was proved using symplectically embedded spheres, and inherent in that method is that the proof only works for restricted $\Omega$. The analogous proof with ECH capacities would include many $\Omega$ whose $c_{X_{\Omega}}$ we currently know nothing about, for example $X_{\Omega} = E(b, 1)$ with $b$ irrational.

On the other hand, [3, Thm. 94], which proves $c_{H_b}$ has no infinite staircase, was proved using ECH capacities using a method that is hard to generalize to other rational values of $b$ where $c_{H_b}$ is unobstructed at the accumulation point but has no infinite staircase (these form the countable family in Thm. 3.3 (iii)). Translating this proof to symplectically embedded spheres may make it more generalizable, and will aid in proving the Cristofaro-Gardiner–Holm–Mandini–Pires conjecture.

**Project 3.5.**

1. Prove the heuristic equivalence above. That is, prove that the ECH cobordism map actually counts genus zero $J$-holomorphic curves in the correct homology classes, as predicted by [25, 10], and extending [3, Prop. 5]. This generalizes [3, Lem. 92].

2. Compute $c_{E(b, 1)}$ for some irrational $b$ (this seems to only be possible using ECH capacities).
(3) Prove the Cristofaro-Gardiner–Holm–Mandini–Pires conjecture, by translating the proof of Thm. 94 to symplectically embedded spheres.

References