

Corrigendum to “Mean action of periodic orbits of area-preserving annulus diffeomorphisms”

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1 Introduction

This corrigendum corrects several mathematical errors in [5]. Consequently, the main result [5, Thm. 1.9] acquires an extra hypothesis: that the annulus diffeomorphisms under consideration must be isotopic, relative to the boundary, to rotation. We have also changed the hypothesis on $\mathcal{V}(\tilde{\psi})$.

We first state the new main theorem, then prove the (very immediate) interpretation of the result as a “zero or infinity” statement. Next, we explain in Remark [1.4] the reason for one of the new hypotheses of Theorem [1.1] from the perspective of embedded contact homology (ECH). Finally, in §[1.2], we list the errors in [5] before embarking upon their corrections in §[2] and §[3]. Throughout this corrigendum we freely use notation set up in [5], as well as all results besides those indicated as erroneous in §[1.2].

1.1 Main theorem

The new main theorem, replacing [5, Thm. 1.9], is

Theorem 1.1. *Let $y_0 \in \mathbb{R}$ and let ψ be an area-preserving diffeomorphism of (A, ω) , with $\tilde{\psi}$ a lift of ψ to \tilde{A} which is translation by $2\pi y_0$ near $\partial\tilde{A}$. Let F denote the flux of $\tilde{\psi}$. Assuming*

$$\mathcal{V}(\tilde{\psi}) < \frac{F \min\{y_0, -y_0 + F\}}{2 \max\{y_0, -y_0 + F\}},$$

or that y_0 is rational, we have

$$\inf \left\{ \frac{\mathcal{A}(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}(\psi) \right\} \leq \mathcal{V}(\tilde{\psi}).$$

By replacing (ψ, y_0) with $(\psi^{-1}, -y_0)$, we update [5, Cor. 1.15] to

Corollary 1.2. *Let $y_0 \in \mathbb{R}$ and let ψ be an area-preserving diffeomorphism of (A, ω) , with $\tilde{\psi}$ a lift of ψ to \tilde{A} which is translation by $2\pi y_0$ near $\partial\tilde{A}$. Let F denote the flux of $\tilde{\psi}$. Assuming*

$$\mathcal{V}(\tilde{\psi}) > \frac{F \max\{y_0, -y_0 + F\}}{2 \min\{y_0, -y_0 + F\}},$$

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or that y_0 is rational, we have

$$\sup \left\{ \frac{\mathcal{A}(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}(\psi) \right\} \geq \mathcal{V}(\tilde{\psi}).$$

A consequence of [5] Thm. 1.9, Cor. 1.15] is the following quantitative criterion for an annulus diffeomorphism to have periodic orbits:

Corollary 1.3. *If ψ is an area-preserving diffeomorphism of (A, ω) whose lift $\tilde{\psi}$ to \tilde{A} is translation by $2\pi y_0$ near $\partial\tilde{A}$ and ψ does not have periodic orbits, then y_0 is irrational and*

$$\mathcal{V}(\psi) = y_0 = \frac{F}{2},$$

where F is the flux of $\tilde{\psi}$.

Proof. If the conclusion does not hold, then ψ satisfies the hypotheses of either Theorem [1.1] or Corollary [1.2]. \square

Corollary [1.3] follows in the same manner as it would using the original [5] Thm. 1.9, Cor. 1.15]. While not appearing in [5], we did explain the conclusions of [5] at the time it appeared by stating Corollary [1.3] in other places, so it is not new.

Remark 1.4. The addition of the hypothesis that ψ must rotate each boundary component by the same amount begs the question of whether or not Theorem [1.1] holds when the boundary rotation amounts are different.

We expect it is true, but that it would require more work on the ECH of toric lens spaces in order to prove, which would go far beyond the computations in [5]. The idea of the proof is to compute the knot filtration on the ECH chain complex using a model contact form with two Reeb orbits having the same rotation numbers as the one constructed from the annulus symplectomorphism (i.e., $1/y_+$ and $1/(-y_- + F)$, where y_{\pm} are the boundary rotation numbers, taking the place of y_0 near each boundary component of A). When $y_+ \neq y_-$, it is not possible to devise a toric contact form (see [1] for inspiration about how to extend the ideas of toric domains to lens spaces) with these fixed boundary rotation numbers as a quotient of the boundary of an ellipsoid. Like ellipsoids, the ECH differential vanishes for the lens spaces studied in [5]; in the case of general $y_+ \neq y_-$, one would need to model the chain complex using the ideas of [4, 2].

Although we do believe the strategy outlined above could work, it would require delving deeper into the ECH moduli spaces than [5] does, and would rely on [7, 6] (written four years after [5] was) and [1] (which is not published). Therefore, we instead restrict ourselves to strengthening the hypotheses of Theorem [1.1], and save the more general case for future work.

Remark 1.5. (i) The new hypothesis implies that $\mathcal{V}(\tilde{\psi}) < F/2$, meaning that [5] Prop. A.1] no longer holds. However, that proposition did not imply that in all other cases, Theorem [1.1] could be recovered from [3] Thm. 1.2], it simply identified one scenario in which the disk theorem does not imply the annulus theorem. Moreover, by generalizing the computation of $f_{\kappa}(0, 0)$ it is possible to show that $f_{\kappa}(r, \theta)$ and $f(\kappa^{-1}(r, \theta))$ differ by a term involving the integral of $\beta/2$, and so a periodic point other than the origin identified by [3] Thm. 1.2] would not necessarily satisfy the conclusion of Theorem [1.1].

(ii) Note that

$$\frac{F \min\{y_0, -y_0 + F\}}{2 \max\{y_0, -y_0 + F\}} \leq \min\{y_0, -y_0 + F\}$$

as $F = \min\{y_0, -y_0 + F\} + \max\{y_0, -y_0 + F\}$.

1.2 Corrections to the proof

Certain sections of [5] (§3 and §5, and parts of §6) require modification, with the most significant changes in §5 and the beginning of §6. The two significant errors are the following:

1. The statement and proof of [5, Prop. 3.1] are incorrect; in particular, in Step 3 of the proof, the contact manifold is misidentified as $L(y_+ - y_- + F, y_+ - y_- + F - 1)$, when in fact it is $L(F, F - 1)$. While we believe the rest of the paper (barring the errors below) is entirely correct when $y_+ = y_-$, several sections (§5.2, §5.3, §6.1, and §6.2) require adjustments to indicate they only apply in this special case.
2. In the proof of [5, Prop. 6.3] we need to show that a Reeb orbit set satisfying certain action and intersection number inequalities is nonempty. We accomplish this by showing that its intersection number with a page, equation (6.12), is positive. However, the original argument is incorrect, as it relies on a function C of N , defined in (6.14), to be uniformly bounded below one as N goes to infinity (the parameter $N \in \mathbb{Z}$ corresponds to the choice $\tilde{\psi}$ of lift of ψ to \tilde{A}). This is not the case.

We have fixed this error by changing our hypothesis on $V(\tilde{\psi})$. See Remark [1.5] and the discussion following Remark [3.2].

We have also identified two less impactful errors:

3. The statement of [5, Prop. 6.1] applies to all contact forms, while the proof only accounts for nondegenerate forms. We simply note here that the extension to the case of degenerate forms follows exactly as in Step 2 of the proof of [3, Prop. 3.1].
4. The original paper used the word “flux” in a nonstandard way, referring to the flux of the map $\tilde{\psi}$ rather than the map ψ . We correct this here. Moreover, our new hypothesis on $\mathcal{V}(\tilde{\psi})$ allows us to decrease our dependence on the relationship between $\tilde{\psi}$ and ψ in the proof of Proposition [3.1].

We correct the first error in [2] and the second error in [3]. Throughout this corrigendum we use the notation of [5].

1.3 Acknowledgements

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2 Changes to §3, §5.2, §5.3, and §6.1

In this section we make the adjustments necessary only due to error #1.

2.1 Correction to [5, Prop. 3.1] and its proof

The correct version of [5, Prop. 3.1], in which we constructed a contact manifold from the mapping torus of the annulus symplectomorphism ψ , is given in Proposition 2.1. First we explain the idea, which highlights in more detail why the original construction was incorrect. Note that in this subsection we are not necessarily assuming $y_+ = y_-$.

The goal is to construct a contact manifold (Y, λ) for which

- the annulus A is a global surface of section for the Reeb flow, with return map ψ ,
- the rotation numbers of the binding orbits are the reciprocals of the values of the action function on the corresponding boundary components of A ,
- the return time is the action function, and
- the contact volume is the Calabi invariant times the symplectic area of A .

These properties are listed as the conclusions of Proposition 2.1, with more precision.

We build Y from the mapping torus of ψ together with a contact form constructed so that its Reeb vector field equals the $[0, 1]$ direction of the mapping torus and the last two conditions above (on return time and contact volume) hold. The next step is to glue solid tori to a neighborhood of the boundary of the mapping torus so that the condition on rotation numbers holds. This is where we see the most significant difference from the situation in [3, Prop. 2.1]. Along the $x = -1$ boundary, the action function no longer equals the boundary rotation number, but involves an extra flux term.

When identifying the monodromy of the open book supporting $\ker \lambda$, we need to compute the return map of a vector field which points in the meridional direction near the binding. This return map will differ from that of the Reeb vector field near a binding component by a twist by the value of action function on the corresponding boundary component of A . Thus near the $x = +1$ boundary component the monodromy simply “untwists” the return map, while near the $x = -1$ boundary component the monodromy untwists the return map but overshoots by the difference between the value of the action function at $x = -1$ and the amount by which ψ rotates along $x = -1$; this difference is F .

In the gluing step of the original proof we introduced coordinates \hat{y} and \check{y} ; we believe these coordinates complicated the original proof unnecessarily in the annulus setting, leading to our confusion on the computation of the monodromy map. We have removed them in the updated proof below.

Proposition 2.1. *Let ψ be an area-preserving diffeomorphism of (A, ω) which is rotation by $2\pi y_{\pm}$ near $\partial_{\pm} A$, whose flux is $F \in \mathbb{Z}$, for which both y_+ and $-y_- + F$ are irrational, and whose action function f is positive. Then there is a contact form $\lambda_{\check{y}}$ on $L(F, F - 1)$ for which:*

1. *An open book decomposition (B_F, P_F) of $L(F, F - 1)$ with abstract open book (A, D_F) is adapted to $\lambda_{\check{y}}$. Let A_0 denote the closure of the zero page. The return time of the Reeb flow from A_0 to A_0 is given by the action function f , and ψ is the return map of $(\lambda_{\check{y}}, B_F, P_F)$.*
2. *The binding orbits have action one, are elliptic, and have rotation numbers $\frac{1}{y_+}$ and $\frac{1}{-y_- + F}$ in the trivializations which have linking number zero with their component of B_F with respect to A_0 .*

3. Let $\{|B_F|\}$ denote the set of components of B_F . There is a bijection $\mathcal{P}(\psi) \cup \{|B_F|\} \rightarrow \mathcal{P}(\lambda_{\tilde{\psi}})$. The symplectic action of the Reeb orbit corresponding to $\gamma \in \mathcal{P}(\psi)$ is $\mathcal{A}(\gamma)$, and its intersection number with the page A_0 is $\ell(\gamma)$.

4. The contact volume satisfies $\text{vol}(L(F, F-1), \lambda_{\tilde{\psi}}) = 2\mathcal{V}(\psi)$.

Proof. Step 1 holds without change, and Steps 4-5 can be replaced with exact analogues. Replace Steps 2-3 with the following:

Step 2: The closed manifold

Consider the oriented coordinates (ρ_+, μ_+, t_+) and (ρ_-, t_-, μ_-) on the solid tori $\mathbb{T}_{\pm} = \mathbb{D}^2(\epsilon_{\pm}) \times (\mathbb{R}/2\pi\mathbb{Z})$, where $\rho_{\pm} \in [0, \epsilon_{\pm}]$ and $\mu_{\pm} \in \mathbb{R}/2\pi\mathbb{Z}$ are coordinates on $\mathbb{D}^2(\epsilon_{\pm})$ and the coordinate on $\mathbb{R}/2\pi\mathbb{Z}$ is $t_{\pm} \in \mathbb{R}/2\pi\mathbb{Z}$. Let $g_{\pm} : M_{\psi} \rightarrow \mathbb{T}_{\pm}$ be given by

$$\begin{aligned} g_+(x, y, \theta) &= (\sqrt{1-x}, 2\pi\theta, y + 2\pi\theta y_+) \\ g_-(x, \theta, y) &= (\sqrt{x+1}, y + 2\pi\theta(y_- - F), 2\pi\theta), \end{aligned}$$

in oriented coordinates on both the domain and target. Because $F \in \mathbb{Z}$, the map g_- is well-defined.

Let Y_{ψ} denote the union of M_{ψ} with the \mathbb{T}_{\pm} s via the g_{\pm} s.

Step 3: Open book decomposition

Denote by B_F the subset of Y_{ψ} where $\{\rho_{\pm} = 0\}$. Let $P_F : Y_{\psi} - B_F \rightarrow S^1$ be given by $(t, z) \mapsto t$. The preimages $P_F^{-1}(t)$ are diffeomorphic to \mathring{A} . We claim that P_F is a projection map for an open book decomposition with page A .

The meridional direction near the component of B_F corresponding to $\partial_{\pm}A$ is given by $\partial_{\mu_{\pm}}$, which extends to M_{ψ} as $-y_+\partial_y + \frac{1}{2\pi}\partial_{\theta}$ near ∂_+A and $(-y_- + F)\partial_y + \frac{1}{2\pi}\partial_{\theta}$ near ∂_-A . The direction ∂_{θ} is transverse to the fibers of P_F . Choose smooth monotone interpolations

- $\delta_+ : [-1, 1] \rightarrow [-y_+, 0]$ with $\delta_+|_{[-1, 1-\epsilon_+^2]} = 0$ and $\delta_+(1) = -y_+$,
- $\delta_- : [-1, 1] \rightarrow [0, -y_- + F]$ with $\delta_-|_{[\epsilon_-^2, -1, 1]} = 0$ and $\delta_-(-1) = -y_- + F$.

Let V be the vector field

$$V = (\delta_+(x) + \delta_-(x))\partial_y + \frac{1}{2\pi}\partial_{\theta},$$

which is transverse to the pages of P_F and equals $\partial_{\mu_{\pm}}$ near B_F .

We claim that the return map of the flow of V from $P_F^{-1}(0)$ to itself is homotopic (relative to ∂A) to the F -fold right-handed Dehn twist D_F . Because the coefficient of ∂_{θ} in V is $\frac{1}{2\pi}$, it takes at least time 2π to send $P^{-1}(0)$ to itself. The return map of the time 2π flow of V near the ∂_+A component of $P^{-1}(0)$ is

$$(x, y, 0) \mapsto (x, y - 2\pi y_+, 1) \sim (x, y, 0),$$

while near the ∂_-A component, the return map is

$$(x, 0, y) \mapsto (x, 1, y + 2\pi(-y_- + F)) \sim (x, 0, y + 2\pi F),$$

where we do not make the simplification $y + 2\pi F \sim y \in \mathbb{R}/2\pi\mathbb{Z}$ to emphasize the F -fold right-handed Dehn twist. \square

Throughout the paper, \tilde{p} should be replaced with F ; below, we discuss only changes to notation, results, and proofs, and leave it to the reader to make the necessary changes to the connecting text.

2.2 Corrections to §5.2

From here on out, we assume $y_+ = y_- = y_0$. The correct version of [5, Lem. 5.5] is the following:

Lemma 2.2. *The rotation numbers of e_{\pm}^F in the trivializations of $\ker \lambda_{\tilde{\psi}}$ which have linking number zero with e_{\pm}^F with respect to their Seifert surfaces are $\frac{F}{y_0} - 1$ and $\frac{F}{-y_0+F} - 1$.*

Proof. Replace \tilde{p} with F in the proof of [5, Lem. 5.5]. □

The model contact forms constructed in [5, Prop. 5.4] and used later to compute the knot filtration only have the correct binding rotation numbers for both binding components when $y_+ = y_-$. In general, we can only expect one of the rotation numbers of e_{\pm} to agree with those of Proposition 2.1. The corrected version is as follows:

Proposition 2.3. *If $\frac{F}{y_0} - 1, \frac{F}{-y_0+F} - 1 \in \mathbb{R} \setminus \mathbb{Q}$, there is a nondegenerate contact form $\lambda_{y_0}^F$ on $L(F, F - 1)$ satisfying*

1. $\ker \lambda_{y_0}^F$ and $\ker \lambda_{\tilde{\psi}}$ are contactomorphic.
2. Under the diffeomorphism of 1., the orbits e_{\pm} of $\lambda_{\tilde{\psi}}$ are both also simple nondegenerate elliptic Reeb orbits for $\lambda_{y_0}^F$, and $\lambda_{y_0}^F$ has no other simple Reeb orbits.
3. (a) The nullhomologous cover e_+^F of e_+ has rotation number $\frac{F}{y_0} - 1$ and as a Reeb orbit of λ_+^F when computed in the trivialization of $\ker \lambda_+^F$ which has linking number zero with e_+^F with respect to its Seifert surface S_+ .
- (b) The nullhomologous cover e_-^F of e_- has rotation number $\frac{F}{-y_0+F} - 1$ as a Reeb orbit of λ_-^F when computed in the trivialization of $\ker \lambda_-^F$ which has linking number zero with e_-^F with respect to its Seifert surface S_- .

Proof. The proof is identical to that of [5, Prop. 5.4], except we define

$$\mathfrak{q}_F^* \lambda_{y_0}^F = \lambda_{(1, b_0)},$$

where

$$b_0 := \frac{y_0}{-y_0 + F}.$$

□

The connecting text in the rest of §5.2 can be read as-is, replacing \tilde{p} with F . We thus obtain a combinatorial chain complex for $ECC_*(L(F, F - 1), \lambda_{y_0}^F, J)$, which we describe in the following way.

Proposition 2.4. *1. The generators of $ECC_*(L(F, F - 1), \lambda_{y_0}^F, J)$ correspond to points (d, m_+) in the second skew quadrant determined by the x -axis and the line $y = Fx$:*

$$(d, m_+) \leftrightarrow e_+^{m_+} e_-^{m_-}, \text{ where } \frac{m_+ - m_-}{F} =: d.$$

2. There is a bijection between generators and $2\mathbb{Z}_{\geq 0}$ given by the order in which a line of slope y_0 moving northwest passes through the points in the second skew quadrant in 1.

2.3 Corrections to §5.3

Note that by simple geometry, the y -coordinate of the y -intercept of the line through (d, m_+) of slope y_0 equals $f_{\pm} \mathcal{F}_{e_{\pm}}(e^{m_+} e^{m_-})$, where $f_+ = y_0$ and $f_- = -y_0 + F$, the values of the action function on $\partial_{\pm} A$. As in [5] §5.3], this can be used to prove the computation of the link filtration on the ECH of $L(F, F - 1)$, which was proved but not stated in [5] Prop. 5.9] in the special case $y_+ = y_- = y_0$.

Proposition 2.5.

$$ECH_{2k}^{\mathcal{F}_{e_+} + \mathcal{F}_{e_-} \leq \ell} \left(L(F, F - 1), \xi_{\tilde{\psi}}, e_+, e_-, \text{rot}(e_+), \text{rot}(e_-) \right) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \ell \geq N_{w(k)} \left(\frac{1}{y_0}, \frac{1}{-y_0 + F} \right) \\ 0 & \text{else} \end{cases}.$$

2.4 Corrections to §6.1

The identification of a Reeb orbit of $\lambda_{\tilde{\psi}}$ satisfying the necessary suite of numerical properties in [5] Prop. 6.1] must be corrected to the following.

Proposition 2.6. *Let λ be a contact form on $L(F, F - 1)$ contactomorphic to the contact form λ_F from [5] Lem. 2.6]. Suppose that both binding components b_{\pm} of the open book decomposition (H_F, Π_F) are elliptic and their nullhomologous covers b_{\pm}^F have rotation numbers equal to those of e_{\pm}^F as in Lemma 2.2]. Then, for all $\epsilon > 0$, for all sufficiently large integers k there is an orbit set α_k not including either b_{\pm} and nonnegative integers $m_{k,\pm}$ for which*

$$I(b_+^{m_{k,+}} \alpha_k b_-^{m_{k,-}}) = 2k$$

$$\mathcal{A}(\alpha_k) \leq \sqrt{2k(\text{vol}(L(F, F - 1), \lambda) + \epsilon)} - m_{k,+} \mathcal{A}(b_+) - m_{k,-} \mathcal{A}(b_-) \quad (2.1)$$

$$\alpha_k \cdot A_0 \geq N_{w(k)} \left(\text{rot}(b_+) + \frac{1}{F}, \text{rot}(b_-) + \frac{1}{F} \right) - m_{k,+} \text{rot}(b_+) - m_{k,-} \text{rot}(b_-). \quad (2.2)$$

Proof. The proof is very similar to that of [5] Prop. 6.1]. We outline the differences here.

The first step, which invokes the approximation of the contact volume by ECH capacities, is identical. Thus we can assume there is some k for which:

- there exists a cycle $x_k \in ECC_{2k}(L(F, F - 1), \lambda, J)$ representing the generator of the group $ECH_{2k}(L(F, F - 1), \ker \lambda_F)$,
- we may write $x_k = \sum_i x_{k_i}$, where each x_{k_i} is an admissible orbit set and the sum is finite,
- and for all i , the action is bounded: $\mathcal{A}(x_{k_i}) \leq \sqrt{2k \text{vol}(L(F, F - 1), \lambda) + \epsilon}$.

Writing $x_{k_i} = b_+^{m_{k_i,+}} \alpha b_-^{m_{k_i,-}}$, where α is an admissible orbit set not including either b_{\pm} , gives us (2.1) for each i .

Because the contact structures $\ker \lambda_{\pm}^F$ and $\ker \lambda$ are contactomorphic (all being contactomorphic to the model $\ker \lambda_F$), and $\text{rot}(b_{\pm}^F) = \text{rot}(e_{\pm}^F)$, Proposition 2.5] shows that there must be some i for which

$$(\mathcal{F}_{b_+} + \mathcal{F}_{b_-})(x_{k_i}) \geq N_{w(k)} \left(\frac{1}{y_0}, \frac{1}{-y_0 + F} \right),$$

from which (2.2) follows as in the original proof. □

3 Corrections to §6.2

In this section we correct error #2 explained in §1.2, keeping in mind the changes put in place by the corrections in the previous section.

The key argument in [5, Prop. 6.3], which transforms the Reeb orbit existence shown in Proposition 2.6 into an annulus periodic orbit existence result with an estimate involving $\mathcal{V}(\tilde{\psi})$, must be corrected to the following:

Proposition 3.1. *Let ψ be an area-preserving diffeomorphism of (A, ω) which is rotation by $2\pi y_0$ near ∂A , whose flux applied to the class of the $(x, 0)$ curve in \tilde{A} is $F \in \mathbb{Z}$, whose action function f is positive, and where y_0 and $-y_0 + F$ are irrational. Further assume*

$$\mathcal{V}(\tilde{\psi}) < \frac{F \min\{y_0, -y_0 + F\}}{2 \max\{y_0, -y_0 + F\}}.$$

Then

$$\inf \left\{ \frac{\mathcal{A}(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathcal{P}(\psi) \right\} \leq \sqrt{\text{hm}(y_0, -y_0 + F)(\mathcal{V}(\tilde{\psi}))}. \quad (3.1)$$

Proof. Note that the hypotheses imply that also $\frac{F}{y_0} - 1$ and $\frac{F}{-y_0 + F} - 1$ are irrational, so we can apply Proposition 2.6.

The proof is the same until the line above [5, (6.13)]. Our goal remains [5, (6.15)]:

$$\frac{m_{k,+}}{y_0} + \frac{m_{k,-}}{-y_0 + F} < \sqrt{\frac{2kF}{y_0(-y_0 + F)} - c_1 k^{1/2} + c_2}. \quad (3.2)$$

When k is large enough, we may ignore the contribution from $-c_1 k^{1/2} + c_2$. Let

$$m = \min\{y_0, -y_0 + F\} \quad \text{and} \quad M = \max\{y_0, -y_0 + F\}.$$

By (2.1) and the fourth conclusion of Proposition 2.1, we have

$$m_{k,+} + m_{k,-} \leq \sqrt{2k(2\mathcal{V}(\tilde{\psi}) + \epsilon)}$$

as the action $\mathcal{A}(\alpha_k)$ is nonnegative. If we choose ϵ small enough, we obtain

$$m_{k,+} + m_{k,-} \leq \sqrt{\frac{2kFm}{M}} \iff \frac{m_{k,+}}{m} + \frac{m_{k,-}}{m} \leq \sqrt{\frac{2kF}{mM}},$$

which is stronger than our goal (3.2). The rest of the proof proceeds as before. \square

Remark 3.2. (i) Note that although we no longer rely on [5, Lem. 6.4] to apply [5, Prop. 6.3] to prove Theorem 1.1, we still need it to apply Proposition 2.1.

(ii) One might wonder why we need to assume

$$\mathcal{V}(\tilde{\psi}) < \frac{F \min\{y_0, -y_0 + F\}}{2 \max\{y_0, -y_0 + F\}}$$

in light of [5, Lem. 6.4] when the left hand side of (3.2) is evidently asymptotic to $1/N$ while the right hand side is asymptotic to $1/\sqrt{N}$ and so eventually larger (here $N \in \mathbb{Z}$ indicates the difference in choices of lift $\tilde{\psi}$ of ψ to \tilde{A}). The problem is that when N increases, the three-manifold and its contact form from Proposition 2.1 both change. Because the actions of the Reeb orbits do not scale uniformly (the binding orbits always have action one while the actions of the other orbits increase by N), we have no control on how the $m_{k,\pm}$ might change with N .

When we use the action bound to replace the $m_{k,\pm}$ with a function of $\mathcal{V}(\tilde{\psi})$, we now have quantities that all vary similarly with N , but they are asymptotic to each other, hence the change in the hypothesis on $\mathcal{V}(\tilde{\psi})$.

The arguments in the rest of the paper may now be applied as written, with Cases 1(b), 2(a)(iii), 2(a)(iv), and 2(b) removed. Note that we also no longer have to lift the requirement $y_+ - y_- \in \mathbb{Z}$, as that is not a hypothesis of Proposition 2.1 (though it is of its analogue [5, Prop. 3.1]).

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