Corrigendum to "Mean action of periodic orbits of area-preserving annulus diffeomorphisms"

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1 Introduction

This corrigendum corrects two mathematical errors in [5]. Consequently, the main result [5, Thm. 1.9] acquires an extra hypothesis: that the annulus diffeomorphisms under consideration must be isotopic, relative to the boundary, to a nonnegative twist. However, [5, Thm 1.9] and its [5, Cor. 1.15] still together provide a (smooth version of a) quantitative interpretation of Franks' result that annulus homeomorphisms have either two or infinitely many periodic orbits.

We next state the new main theorem, then prove the (very immediate) interpretation of the results as a "zero or infinity" statement. Next, we explain in Remark 1.4 the reason why the new hypothesis of Theorem 1.1 is not surprising. Finally, in §1.2, we explain the two errors in [5] before embarking upon their corrections in §2 and §3. Throughout this corrigendum we freely use notation set up in [5], as well as all results besides those indicated as erroneous in §1.2.

1.1 Main theorem

The new main theorem, replacing [5, Thm. 1.9], is

Theorem 1.1. *Let $y_\pm \in \mathbb{R}$ with $y_- \geq y_+$. Let ψ be an area-preserving diffeomorphism of (A, ω), with $\tilde{\psi}$ a lift of ψ to $\tilde{A}$ which is translation by $2\pi y_+$ near $\{1\} \times \mathbb{R}$ and by $2\pi y_-$ near $\{-1\} \times \mathbb{R}$. Let F denote the flux of ψ. Assuming*

$$\mathcal{V}(\tilde{\psi}) < \max\{y_+, -y_- + F\},$$

or that one of $y_\pm$ is rational, we have

$$\inf\left\{ \frac{\mathcal{A}(\gamma)}{\ell(\gamma)} \middle| \gamma \in \mathcal{P}(\psi) \right\} \leq \mathcal{V}(\tilde{\psi}).$$

By replacing (ψ, y_+) with $(\psi^{-1}, -y_+)$ (notice that this changes the order of the boundary rotation numbers), we update [5, Cor. 1.15] to

Corollary 1.2. *Let $y_\pm \in \mathbb{R}$ with $y_- \leq y_+$. Let ψ be an area-preserving diffeomorphism of (A, ω), with $\tilde{\psi}$ a lift of ψ to $\tilde{A}$ which is translation by $2\pi y_+$ near $\{1\} \times \mathbb{R}$ and by $2\pi y_-$ near $\{-1\} \times \mathbb{R}$. Let F denote the flux of ψ. Assuming*

$$\mathcal{V}(\tilde{\psi}) > \min\{y_+, -y_- + F\},$$

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or that one of $y_\pm$ is rational, we have

$$\sup\left\{\frac{\mathcal{A}(\gamma)}{\ell(\gamma)}\middle|\gamma\in\mathcal{P}(\psi)\right\}\geq\mathcal{V}(\tilde{\psi}).$$

A consequence of [5, Thm. 1.9, Cor. 1.15] is the following quantitative criterion for an annulus diffeomorphism to have periodic orbits:

Corollary 1.3. *If ψ is an area-preserving diffeomorphism of (A,ω) which is translation by $2\pi y_\pm$ near $\{\pm 1\}\times\mathbb{R}$ and ψ does not have periodic orbits, then both $y_\pm$ are irrational and*

$$-y_- + F = \mathcal{V}(\psi) = y_+.$$

Proof. If the conclusion does not hold, then ψ satisfies the hypotheses of either Theorem 1.1 or Corollary 1.2. □

Corollary 1.3 follows in the same manner as it would using the original [5, Thm. 1.9, Cor. 1.15]. While not appearing in [5], we did explain the conclusions of [5] at the time it appeared by stating Corollary 1.3, so it is not new.

Remark 1.4. The additional hypothesis which makes the corrected Theorem 1.1 weaker than the originally claimed [5, Thm. 1.9] is that $y_-\geq y_+$. This begs the question of whether or not Theorem 1.1 holds when $y_-<y_+$.

With the benefit of hindsight, we suspect that it is not possible to use the methods of [5] to study the case of $y_-<y_+$. The idea of the proof is to compute the knot filtration on the ECH chain complex using a model contact form that has the same rotation numbers as the one constructed from the annulus symplectomorphism (these are roughly proportionate to $1/y_+$ and $1/y_-$). When $y_-\geq y_+$, it is possible to devise a "convex toric" contact form (see [1] for inspiration about how to extend the ideas of convex and concave toric domains to lens spaces) with these fixed boundary rotation numbers. However, if $y_-<y_+$, this is not possible: the model must be "concave."

The problem then becomes the fact that the ECH differential in the convex case generally makes lattice paths flatter and shorter (decreasing the knot filtration and making it possible to estimate), while in the concave case it does not. See [4] for an explanation of the convex toric differential in the case of T^3 and [2] for the concave differential in the case of S^3 ([1] explains the differential on general toric contact manifolds, a class which includes lens spaces).

Note that although we do believe the strategy outlined above could work, it would require delving deeper into the ECH moduli spaces than [5] does, and would rely on [7, 6] (written four years after [5] was) and [1] (which is not published). Therefore, we instead combine computations from two irrational lens spaces, constructed simply as quotients of irrational ellipsoids, as the estimates we obtain in this way are good enough for our purposes.

Finally, we do note that it might be possible to relax the new hypothesis $y_-\geq y_+$ if the hypothesis $\mathcal{V}(\tilde{\psi})\leq\max\{y_+,-y_-+F\}$ were strengthened, or additional perturbations were considered in the final proof (where an estimate using the the harmonic mean of the boundary values of the action function is replaced with an estimate involving only the Calabi invariant). In order to avoid going too far beyond the scope of the original paper, we do not attempt this here.

1.2 Corrections to the proof

Certain sections of [5] (§3 and §5, and parts of §6) require modification, with the most significant changes in §5 and the beginning of §6. The errors are the following:

1. The statement and proof of [5, Prop. 3.1] are incorrect; in particular, in Step 3 of the proof, the contact manifold is misidentified as $L(y_+ - y_- + F, y_+ - y_- + F - 1)$, when in fact it is $L(F, F-1)$. While the rest of the paper (barring the second error below) is entirely correct as far as we know, several sections (§5.2, §5.3, §6.1, and §6.2) require adjustments as they now only apply in the case when $y_+ - y_- + F = F$, i.e., when $y_+ = y_-$.

2. In the proof of [5, Prop. 6.3] we need to show that a Reeb orbit satisfying certain action and intersection number inequalities is nonempty. We accomplish this by showing that its action is positive in equation (6.16). However, the original argument is incorrect, as it relies on a function C of N, defined in (6.14), to be uniformly bounded below one as N goes to infinity. This is not the case.

   Our new method bypasses Lemma 6.2 and instead uses the more straightforward [3, Lem. 3.2]. The main new feature is that instead of using the Reeb dynamics of a single lens space to capture the dynamics of an annulus map, we use two different lens spaces.

   One feature of the new proof is that it no longer requires the hypothesis $\mathcal{V}(\tilde{\psi}) < \max\{y_+, -y_- + F\}$. This bound on $\mathcal{V}(\tilde{\psi})$ now comes into play only in the final stage, when improving (3.1) to the conclusion of Theorem 1.1. Thus it is reasonable to ask whether or not Theorem 1.1 could be extended to a class of annulus maps with a less restrictive relationship between $\mathcal{V}(\tilde{\psi})$ and $\psi|_{\partial A}$; we do not attempt to pursue this train of thought here.

We correct the first error in §2 and the second error in §3. Throughout this corrigendum we use the notation of [5].

1.3 Acknowledgements

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2 Changes to §3, §5.2, §5.3, and §6.1

In this section we make the adjustments necessary only due to error #1.

2.1 Correction to Proposition 3.1 and its proof

The correct version of [5, Propositon 3.1], in which we constructed a contact manifold from the mapping torus of the annulus symplectomorphism ψ, is given in Proposition 2.1. First we explain the idea, which highlights in more detail why the original construction was incorrect.

The goal is to construct a contact manifold (Y, λ) for which

- the annulus A is a global surface of section for the Reeb flow, with return map ψ,
- the rotation numbers of the binding orbits are the reciprocals of the values of the action function on the corresponding boundary components of A,
- the return time is the action function, and

- the contact volume is the Calabi invariant times the symplectic area of A.

These properties are listed as the conclusions of Proposition 2.1, with more precision.

We build Y from the mapping torus of ψ together with a contact form constructed so that its Reeb vector field equals the $[0,1]$ direction of the mapping torus and the last two conditions above (on return time and contact volume) hold. The next step is to glue solid tori to a neighborhood of the boundary of the mapping torus so that the condition on rotation numbers holds. This is where we see the most significant difference from the situation in [3, Prop. 2.1]. Along the $x=-1$ boundary, the action function no longer equals the boundary rotation number, but involves an extra flux term.

When identifying the monodromy of the open book supporting $\ker\lambda$, we need to compute the return map of a vector field which points in the meridional direction near the binding. This return map will differ from that of the Reeb vector field near a binding component by a twist by the value of action function on the corresponding boundary component of A. Thus near the $x=+1$ boundary component the monodromy simply "untwists" the return map, while near the $x=-1$ boundary component the monodromy untwists the return map but overshoots by the difference between the value of the action function at $x=-1$ and the amount by which ψ rotates along $x=-1$; this difference is F.

In the gluing step of the original proof we introduced coordinates $\hat{y}$ and $\check{y}$; we believe these coordinates complicated the original proof unnecessarily in the annulus setting, leading to our confusion on the computation of the monodromy map. We have removed them in the updated proof below.

Proposition 2.1. *Let ψ be an area-preserving diffeomorphism of (A,ω) which is rotation by $2\pi y_\pm$ near $\partial_\pm A$, whose flux is $F\in\mathbb{Z}$, for which both y_+ and $-y_-+F$ are irrational, and whose action function f is positive. Then there is a contact form $\lambda_{\tilde{\psi}}$ on $L(F,F-1)$ for which:*

1. *An open book decomposition (B_F,P_F) of $L(F,F-1)$ with abstract open book (A,D_F) is adapted to $\lambda_{\tilde{\psi}}$. Let A_0 denote the closure of the zero page. The return time of the Reeb flow from A_0 to A_0 is given by the action function f, and ψ is the return map of $(\lambda_{\tilde{\psi}},B_F,P_F)$.*

2. *The binding orbits have action one, are elliptic, and have rotation numbers $\frac{1}{y_+}$ and $\frac{1}{-y_-+F}$ in the trivializations which have linking number zero with their component of B_F with respect to A_0.*

3. *Let $\{|B_F|\}$ denote the set of components of B_F. There is a bijection $\mathcal{P}(\psi)\cup\{|B_F|\}\to\mathcal{P}(\lambda_{\tilde{\psi}})$. The symplectic action of the Reeb orbit corresponding to $\gamma\in\mathcal{P}(\psi)$ is $\mathcal{A}(\gamma)$, and its intersection number with the page A_0 is $\ell(\gamma)$.*

4. *The contact volume satisfies $\operatorname{vol}(L(F,F-1),\lambda_{\tilde{\psi}})=2\mathcal{V}(\psi)$.*

Proof. Step 1 holds without change, and Steps 4-5 can be replaced with exact analogues. Replace Steps 2-3 with the following:

Step 2: The closed manifold

Consider the oriented coordinates (ρ_+,μ_+,t_+) and (ρ_-,t_-,μ_-) on the solid tori $\mathbb{T}_\pm=\mathbb{D}^2(\epsilon_\pm)\times(\mathbb{R}/2\pi\mathbb{Z})$, where $\rho_\pm\in[0,\epsilon_\pm]$ and $\mu_\pm\in\mathbb{R}/2\pi\mathbb{Z}$ are coordinates on $\mathbb{D}^2(\epsilon_\pm)$ and the coordinate on

$\mathbb{R}/2\pi\mathbb{Z}$ is $t_\pm \in \mathbb{R}/2\pi\mathbb{Z}$. Let $g_\pm : \mathring{M}_\psi \to \mathbb{T}_\pm$ be given by

$$g_+(x,y,\theta) = \left(\sqrt{1-x}, 2\pi\theta, y + 2\pi\theta y_+\right)$$
$$g_-(x,\theta,y) = \left(\sqrt{x+1}, y + 2\pi\theta(y_- - F), 2\pi\theta\right),$$

in oriented coordinates on both the domain and target. Because $F \in \mathbb{Z}$, the map g_- is well-defined.

Let Y_ψ denote the union of $\mathring{M}_\psi$ with the $\mathbb{T}_\pm$s via the $g_\pm$s.

Step 3: Open book decomposition

Denote by B_F the subset of Y_ψ where $\{\rho_\pm = 0\}$. Let $P_F : Y_\psi - B_F \to S^1$ be given by $(t,z) \mapsto t$. The preimages $P_F^{-1}(t)$ are diffeomorphic to $\mathring{A}$. We claim that P_F is a projection map for an open book decomposition with page A.

The meridional direction near the component of B_F corresponding to $\partial_\pm A$ is given by $\partial_{\mu_\pm}$, which extends to $\mathring{M}_\psi$ as $-y_+\partial_y + \frac{1}{2\pi}\partial_\theta$ near $\partial_+ A$ and $(-y_- + F)\partial_y + \frac{1}{2\pi}\partial_\theta$ near $\partial_- A$. The direction ∂_θ is transverse to the fibers of P_F. Choose smooth monotone interpolations

- $\delta_+ : [-1,1] \to [-y_+, 0]$ with $\delta_+|_{[-1,1-\epsilon_+^2]} = 0$ and $\delta_+(1) = -y_+$,
- $\delta_- : [-1,1] \to [0, -y_- + F]$ with $\delta_-|_{[\epsilon_-^2-1,1]} = 0$ and $\delta_-(-1) = -y_- + F$.

Let V be the vector field

$$V = (\delta_+(x) + \delta_-(x))\partial_y + \frac{1}{2\pi}\partial_\theta,$$

which is transverse to the pages of P_F and equals $\partial_{\mu_\pm}$ near B_F.

We claim that the return map of the flow of V from $P_F^{-1}(0)$ to itself is homotopic (relative to ∂A) to the F-fold right-handed Dehn twist D_F. Because the coefficient of ∂_θ in V is $\frac{1}{2\pi}$, it takes at least time 2π to send $P^{-1}(0)$ to itself. The return map of the time 2π flow of V near the $\partial_+ A$ component of $P^{-1}(0)$ is

$$(x,y,0) \mapsto (x, y - 2\pi y_+, 1) \sim (x,y,0),$$

while near the $\partial_- A$ component, the return map is

$$(x,0,y) \mapsto (x,1,y + 2\pi(-y_- + F)) \sim (x,0,y+2\pi F),$$

where we do not make the simplification $y + 2\pi F \sim y \in \mathbb{R}/2\pi\mathbb{Z}$ to emphasize the F-fold right-handed Dehn twist. ☐

Throughout the paper, $\tilde{p}$ should be replaced with F; below, we discuss only changes to notation, results, and proofs, and leave it to the reader to make the necessary changes to the connecting text.

2.2 Corrections to §5.2

The correct version of [5, Lem. 5.5] is the following:

Lemma 2.2. *The rotation numbers of $e_\pm^F$ in the trivializations of $\ker \lambda_{\tilde{\psi}}$ which have linking number zero with $e_\pm^F$ with respect to their Seifert surfaces are $\frac{F}{y_+} - 1$ and $\frac{F}{-y_- + F} - 1$.*

Proof. Replace $\tilde{p}$ with F in the proof of [5, Lem. 5.5]. ☐

The model contact forms constructed in [5, Prop. 5.4] and used later to compute the knot filtration only have the correct binding rotation numbers for both binding components when $y_+ = y_-$. In general, we can only expect one of the rotation numbers of $e_\pm$ to agree with those of Proposition 2.1. The corrected version, where we construct two contact forms on $L(F, F-1)$, each of which has the correct binding rotation number along only one component of the binding, is as follows:

Proposition 2.3. *If $\frac{F}{y_+} - 1, \frac{F}{-y_- + F} - 1 \in \mathbb{R} \setminus \mathbb{Q}$, there are nondegenerate contact forms $\lambda_\pm^F$ on $L(F, F-1)$ satisfying*

1. $\ker \lambda_\pm^F$ *and* $\ker \lambda_{\tilde{\psi}}$ *are contactomorphic.*
2. *Under the diffeomorphism of 1., the orbits $e_\pm$ of $\lambda_{\tilde{\psi}}$ are both also simple nondegenerate elliptic Reeb orbits for $\lambda_\pm^F$, and $\lambda_\pm^F$ have no other simple Reeb orbits.*
3. (a) *The nullhomologous cover e_+^F of e_+ has rotation number $\frac{F}{y_+} - 1$ and as a Reeb orbit of λ_+^F when computed in the trivialization of* $\ker \lambda_+^F$ *which has linking number zero with e_+^F with respect to its Seifert surface S_+.*
   (b) *The nullhomologous cover e_-^F of e_- has rotation number $\frac{F}{-y_- + F} - 1$ as a Reeb orbit of λ_-^F when computed in the trivialization of* $\ker \lambda_-^F$ *which has linking number zero with e_-^F with respect to its Seifert surface S_-.*

Proof. The proof is identical to that of [5, Prop. 5.4], except we define

$$\mathfrak{q}_F^* \lambda_\pm^F = \lambda_{(a_\pm, b_\pm)},$$

where

$$a_+ = F - y_+, b_+ = y_+, a_- = F - (-y_- + F) = y_-, b_- = F - y_-.$$

□

The connecting text in the rest of §5.2 can be read as-is, replacing $\tilde{p}$ with F and doubling each result or discussion to apply to both $\lambda_\pm^F$. We thus obtain a combinatorial chain complex for $ECC_*(L(F, F-1), \lambda_\pm^F, J)$, which we describe in the following way.

Proposition 2.4. 1. *The generators of $ECC_*(L(F, F-1), \lambda_\pm^F, J)$ correspond to points (d, m_+) in the second skew quadrant determined by the x-axis and the line $y = Fx$:*

$$(d, m_+) \leftrightarrow e_+^{m_+} e_-^{m_-}, \text{ where } \frac{m_+ - m_-}{F} =: d.$$

2. *There is a bijection between generators and $2\mathbb{Z}_{\geq 0}$ given by, in the case of λ_+^F, the order in which a line of slope y_+ moving northwest passes through the points in the second skew quadrant in 1.; the bijection in the case of λ_-^F is given by a line of slope y_-.*

2.3 Corrections to §5.3

Note that by simple geometry, the y-coordinate of the y-intercept of the line through (d, m_+) of slope $y_\pm$ equals $f_\pm \mathcal{F}_{e_\pm}(e^{m_+}e^{m_-})$, where $f_+ = y_+$ and $f_- = -y_- + F$, the values of the action function on $\partial_\pm A$. This proves the relevant version of the computation of the knot filtration on the ECH of $L(F, F-1)$, which was originally computed in [5, Prop. 5.9] in the special case $y_+ = y_-$.

Proposition 2.5.

$$ECH_{2k}^{\mathcal{F}_{e_+}\leq \ell}\left(L(F,F-1), \xi_{\tilde{\psi}}, e_+, \frac{1}{y_+} - \frac{1}{F}\right) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \ell \geq N_{w_+^F(k)}\left(\frac{1}{y_+} - \frac{1}{F}, \frac{1}{F}\right) \\ 0 & \text{else} \end{cases}$$

$$ECH_{2k}^{\mathcal{F}_{e_-}\leq \ell}\left(L(F,F-1), \xi_{\tilde{\psi}}, e_-, \frac{1}{-y_- + F} - \frac{1}{F}\right) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \ell \geq N_{w_-^F(k)}\left(\frac{1}{F}, \frac{1}{-y_-+F} - \frac{1}{F}\right) \\ 0 & \text{else} \end{cases}.$$

Here $w_\pm^F(k)$ are defined so that $N_{w_\pm^F(k)}(a,b)$ is the k^{th} largest of the sequence of nonnegative integer linear combinations $m_+a + m_-b$ where $m_+ - m_-$ is divisible by F; notice that therefore $w_\pm^F$ also depends on a, b, but we omit this from the notation.

2.4 Corrections to §6.1

The identification of a Reeb orbit of $\lambda_{\tilde{\psi}}$ satisfying the necessary suite of numerical properties in [5, Prop. 6.1] must be corrected to the following.

Proposition 2.6. *Let λ be a contact form on $L(F, F-1)$ contactomorphic to the contact form λ_F from [5, Lem. 2.6]. Suppose that both binding components $b_\pm$ of the open book decomposition (H_F, Π_F) are elliptic and their nullhomologous covers $b_\pm^F$ have rotation numbers equal to those of $e_\pm^F$ as in Lemma 2.2. Then, for all $\epsilon > 0$, for all sufficiently large integers k there is an orbit set α_k not including either $b_\pm$ and nonnegative integers $m_{k,\pm}$ for which*

$$I(b_+^{m_{k,+}}\alpha_k b_-^{m_{k,-}}) = 2k$$

$$\mathcal{A}(\alpha_k) \leq \sqrt{2k(\mathrm{vol}(L(F,F-1),\lambda) + \epsilon)} - m_{k,+}\mathcal{A}(b_+) - m_{k,-}\mathcal{A}(b_-) \tag{2.1}$$

$$\alpha_k \cdot A_0 \geq N_{w_+^F(k)}\left(\mathrm{rot}(b_+), \frac{1}{F}\right) + N_{w_-^F(k)}\left(\frac{1}{F}, \mathrm{rot}(b_-)\right) - m_{k,+}\,\mathrm{rot}(b_+) - m_{k,-}\,\mathrm{rot}(b_-). \tag{2.2}$$

Proof. The proof is very similar to that of [5, Prop. 6.1]. We outline the differences here.

The first step, which invokes the approximation of the contact volume by ECH capacities, is identical. Thus we can assume there is some k for which:

- there exists a cycle $x_k \in ECC_{2k}(L(F,F-1), \lambda, J)$ representing the generator of the group $ECH_{2k}(L(F,F-1), \ker \lambda_F)$,
- we may write $x_k = \sum_i x_{k_i}$, where each x_{k_i} is an admissible orbit set and the sum is finite,
- and for all i, the action is bounded: $\mathcal{A}(x_{k_i}) \leq \sqrt{2k\,\mathrm{vol}(L(F,F-1),\lambda) + \epsilon}$.

Writing $x_{k_i} = b_+^{m_{k_i,+}}\alpha b_-^{m_{k_i,-}}$, where α is an admissible orbit set not including either $b_\pm$, gives us (2.1) for each i.

Because the contact structures $\ker \lambda_\pm^F$ and $\ker\lambda$ are contactomorphic (all being contactomorphic to the model $\ker \lambda_F$), and $\mathrm{rot}(b_\pm^F) = \mathrm{rot}(e_\pm^F)$, Proposition 2.5 shows that there must be some i for which

$$\left(\mathcal{F}_{b_+} + \mathcal{F}_{b_-}\right)(x_{k_i}) \geq N_{w_+^F(k)}\left(\frac{1}{y_+} - \frac{1}{F}, \frac{1}{F}\right) + N_{w_-^F(k)}\left(\frac{1}{F}, \frac{1}{-y_- + F} - \frac{1}{F}\right),$$

from which (2.2 follows as in the original proof.

We further elucidate the application of Proposition 2.5. The idea is that $\mathcal{F}_{b_+} + \mathcal{F}_{b_-}$ is also a filtration on ECC, and the filtered homology is also an invariant of the contact structure and pair of rotation numbers, as is explained in [5, Thm. 5.2]. The lower bound on the sum filtration $\mathcal{F}_{b_+} + \mathcal{F}_{b_-}$ then arises from the lower bounds on each part obtained from the direct computation using the model forms $\lambda_\pm^F$ in Proposition 2.5. □

3 Corrections to §6.2

In this section we correct error #2 explained in §1.2, keeping in mind the changes put in place by the previous corrections.

We first state an additional lemma used to simplify the functions $N_{w_\pm^F(k)}$.

Lemma 3.1. *For infinitely many values of k, we have*

$$N_{w_\pm^F(k)}(a, 1/F) \geq N_k(a, 1).$$

Proof. If $N_{w_\pm^F(k)}(a, 1/F) = m_+ a + m_-/F$ and m_- is divisible by F, then $m_+a + m_-/F$ appears in the sequence $N(a, 1)$, and is at least the k^{th} term since each term in the $N_{w_\pm^F}(a, 1/F)$ sequence with m_- decreased by a multiple of F appears as a term in the $N(a, 1)$ sequence, and there are exactly $k - 1$ of these. □

We omit [5, Lem. 6.2], which was an unnecessary estimate. The key argument in [5, Prop. 6.3], which transforms the Reeb orbit existence shown in Proposition 2.6 into an annulus periodic orbit existence result with an estimate involving $\mathcal{V}(\tilde{\psi})$, must be corrected to the following:

Proposition 3.2. *Let ψ be an area-preserving diffeomorphism of (A, ω) which is rotation by $2\pi y_\pm$ near $\partial_\pm A$, whose flux applied to the class of the $(x, 0)$ curve is $F \in \mathbb{Z}$, whose action function f is positive, and where y_+ and $-y_- + F$ are irrational and where $y_- \geq y_+$.*

Let $\mathcal{A}_N$ denote the total action computed with $f_{(\psi, y_+ + N, \beta)}$ rather than with $f_{(\psi, y_+, \beta)}$. For all large enough integers N,

$$\inf\left\{\frac{\mathcal{A}_N(\gamma)}{\ell(\gamma)}\middle|\gamma \in \mathcal{P}(\psi)\right\} \leq \sqrt{\mathrm{hm}(y_+ + N, -y_- + F + N)(\mathcal{V}(\tilde{\psi}) + N)}. \tag{3.1}$$

Proof. Note that the hypotheses imply that also $\frac{F}{y_+} - 1$ and $\frac{F}{-y_- + F} - 1$ are irrational, so we can apply Proposition 2.6.

The proof is the same until we apply the conclusion of [5, Lem. 6.2] to show that the lower bound on the right hand side of (2.2) is greater than zero (which we need in order to know that $\alpha_k \neq \emptyset$); instead we use [3, Lem. 3.2], whose conclusion is that for some constant c,

$$N_k(a, b) \geq \sqrt{2abk - ck^{\frac{1}{2}}}. \tag{3.2}$$

Conclusion (2.2) of Proposition 2.6 is equivalent, by Lemma 2.2, to

$$\alpha_k \cdot A_0 \geq N_{w_+^F(k)}\left(\frac{1}{y_+} - \frac{1}{F}, \frac{1}{F}\right) + N_{w_-^F(k)}\left(\frac{1}{F}, \frac{1}{-y_- + F} - \frac{1}{F}\right) - m_{k,+}\left(\frac{1}{y_+} - \frac{1}{F}\right) - m_{k,-}\left(\frac{1}{-y_- + F} - \frac{1}{F}\right). \tag{3.3}$$

We assume k is very large and use Lemma 3.1 in the first line and (3.2) in the second to obtain the following lower bound on the sum of the $N_{w_\pm^F(k)}$ terms:

$$\begin{aligned} N_{w_+^F(k)}\left(\frac{1}{y_+} - \frac{1}{F}, \frac{1}{F}\right) + N_{w_-^F(k)}\left(\frac{1}{F}, \frac{1}{-y_- + F} - \frac{1}{F}\right) &\geq N_k\left(\frac{1}{y_+} - \frac{1}{F}, 1\right) + N_k\left(1, \frac{1}{-y_- + F} - \frac{1}{F}\right) \\ &\geq \sqrt{2\left(\frac{1}{y_+} - \frac{1}{F}\right)k - ck^{\frac{1}{2}}} + \sqrt{2\left(\frac{1}{-y_- + F} - \frac{1}{F}\right)k - c'k^{\frac{1}{2}}}. \end{aligned}$$

We use the fact that $\mathcal{A}(\alpha_k) \geq 0$ in (2.1) and $\mathrm{vol} = 2\mathcal{V}$ in Proposition 2.1 4. to obtain the bound

$$m_{k,+} + m_{k,-} \leq \sqrt{4k\mathcal{V}(\tilde{\psi})}.$$

Set $m = \min\{y_+, -y_- + F\}$ and $M = \max\{y_+, -y_- + F\}$. Using the lower bound on the sum of the $N_{w_\pm^F(k)}$ terms and the upper bound on the sum of the $m_{k,\pm}$ terms, showing that the right hand side of (3.3) is strictly positive will follow when k is large enough (so that the $k^{\frac{1}{2}}$ terms do not contribute significantly) from showing

$$\sqrt{2\left(\frac{1}{M} - \frac{1}{F}\right)k} + \sqrt{2\left(\frac{1}{m} - \frac{1}{F}\right)k} > \sqrt{4k\mathcal{V}(\tilde{\psi})}\left(\frac{1}{m} - \frac{1}{F}\right). \tag{3.4}$$

Rewrite $1/M - 1/F = (F - M)/(FM)$ and similarly rewrite $1/m - 1/F$, and replace $m, M, \mathcal{V}(\tilde{\psi})$, and F with $m + N, M + N, \mathcal{V}(\tilde{\psi}) + N$, and $F + 2N$. Considering only the leading order terms, equation (3.4) follows from

$$\sqrt{\frac{N}{2N^2}} + \sqrt{\frac{N}{2N^2}} > \sqrt{2N} \cdot \frac{N}{2N^2},$$

which is elementary. Therefore $\alpha_k \neq \emptyset$.

Combining (2.1) and (2.2) for the corresponding orbit set, reducing to a single orbit γ as at the start of the original proof of [5, Prop. 6.1], and using Proposition 2.1 4. to replace vol with $\mathcal{V}(\tilde{\psi})$, we obtain an upper bound on $\mathcal{A}_N(\gamma)/\ell(\gamma)$ of

$$\frac{\sqrt{2k(2(\mathcal{V}(\tilde{\psi}) + N) + \epsilon)} - m_{k,+} - m_{k,-}}{\sqrt{2k\left(\frac{1}{y_+ + N} - \frac{1}{F+2N}\right)} + \sqrt{2k\left(\frac{1}{-y_- + F + N} - \frac{1}{F+2N}\right)} - m_{k,+}\left(\frac{1}{y_+ + N} - \frac{1}{F+2N}\right) - m_{k,-}\left(\frac{1}{-y_- + F + N} - \frac{1}{F+2N}\right)}.$$

By the same logic as in the original proof, this is maximized when $m_{k,\pm} = 0$, therefore by simplifying we have

$$\frac{\mathcal{A}_N(\gamma)}{\ell(\gamma)} \leq \frac{\sqrt{2(\mathcal{V}(\tilde{\psi}) + N)}}{\sqrt{\frac{1}{y_+ + N} - \frac{1}{F+2N}} + \sqrt{\frac{1}{-y_- + F + N} - \frac{1}{F+2N}}}. \tag{3.5}$$

To show that the right hand side of (3.5) is at most the right hand side of (3.1), we must show:

$$\frac{\sqrt{2}}{\sqrt{\frac{1}{m+N}-\frac{1}{F+2N}}+\sqrt{\frac{1}{M+N}-\frac{1}{F+2N}}} \leq \sqrt{\frac{2}{\frac{1}{m+N}+\frac{1}{M+N}}}$$

$$\sqrt{\left(\frac{1}{m+N}-\frac{1}{F+2N}\right)\left(\frac{1}{M+N}-\frac{1}{F+2N}\right)} \geq \frac{1}{F+2N}$$

$$\frac{1}{(m+N)(M+N)}-\frac{1}{(m+N)(F+2N)}-\frac{1}{(M+N)(F+2N)} \geq 0$$

$$F+2N-(M+N)-(m+N) \geq 0.$$

Thus the conclusion follows when $F \geq M+m$. Notice

$$F \geq M+m \Leftrightarrow F \geq y_+ - y_- + F \Leftrightarrow y_- \geq y_+,$$

which was one of our hypotheses. □

The arguments in the rest of the paper may now be applied as written. Note that we no longer have to worry about lifting the requirement $y_+ - y_- \in \mathbb{Z}$, as that is not a hypothesis of Proposition 2.1 (though it is of its analogue [5, Prop. 3.1]).

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