1 Topology and dynamics in symplectic and contact geometry

My research is in symplectic and contact geometry, focusing on connections with low-dimensional topology, dynamics, and complex geometry. Symplectic and contact geometry occupy an intermediate position between smooth topology and complex geometry. On the smooth side, symplectic and contact manifolds have no local invariants such as curvature, and depending on the manifold, their automorphisms can be what we call "flexible," preserving only volume. Yet many symplectic/contact phenomena are "rigid," i.e., determined globally by local behavior just as holomorphic functions are. Rigidity arises because minimal "J-holomorphic" submanifolds are the source of the major invariants. Exploiting the flexibility-rigidity spectrum has been especially productive in low dimensions: see Gompf's topological constructions of symplectic four-manifolds as evidence of flexibility (e.g. in [Gom95, Gom98]) and Taubes' characterization of the Seiberg-Witten invariants via symplectic submanifolds in [Tau94, Tau95, Tau96] as emblematic of rigidity.

The unifying theme of my work is investigating the smooth-symplectic-complex interplay in dimensions two and four, with a strong emphasis on computation. In the intermediate dimension three, I study the relationship between contact geometry and topology. In this research statement I will describe my recent work, following three major threads organized by dimension, as follows.

- ***Research program 1:*** *4D symplectic embeddings and the toric staircases conjecture (see §2).* In [MMW22, MPW23] I proved a foundational case of a conjecture of Cristofaro-Gardiner–Holm–Mandini–Pires that posits the symplectic embeddings into a 4D toric variety are highly controlled by its algebraic geometry and number theory. This work is ongoing.
- ***Research program 2:*** *contact invariants and 3D topology (see §3).* There is a rich two way relationship between contact forms and topology in dimension three. In one direction, I use topological tools such as open books, knot cobordisms, and circle bundles to compute contact invariants (see my joint work [Wei21, NW20, NW23a, NW23b, RWY] and future work in §3.4.1). Conversely, it is expected that the algebraic structure of these invariants characterizes the smooth topology of the underlying manifold, and in this direction I outline a long term project towards the L-space conjecture in §3.4.2.
- ***Research program 3:*** *2D dynamics via contact geometry (see §4).* In two dimensions, the smooth-symplectic-complex hierarchy collapses. Recently, focus has turned to applications in dynamics. In [Wei21, NW23b] I prove quantitative existence for periodic orbits of area-preserving surface maps with checkable hypotheses.

I have taken a broad perspective: my work ranges from theoretical foundations to writing computer programs. This breadth is due to a fundamental tension intrisic to symplectic and contact geometry. Invariants are built from counts of "J-holomorphic curves," which are solutions to a nonlinear Cauchy-Riemann equation. The counts are defined for generic data, but it is only possible to actually compute the numbers in highly symmetric, non-generic settings. Bridging the gap requires that I revisit the invariants' foundations, while computations require combinatorics.

One consequence of taking my approach is that I have built an ever-growing list of projects suitable for a range of students, from undergraduate through the Ph.D. level. These are described concretely in each section. Furthermore, I have written two papers with undergraduates.

2 Research program 1: 4D symplectic embeddings and the toric staircases conjecture

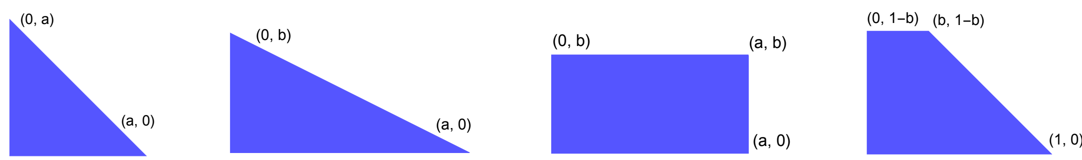
A **symplectic form** is a closed smooth nondegenerate two-form ω on a $2n$-dimensional manifold. Its **volume form** is $\omega^{\wedge n}$. A **symplectic embedding** is an embedding, denoted $X \overset{s}{\hookrightarrow} X'$, which identifies the symplectic forms. Symplectic embeddings are at the heart of the rigidity-flexibility dichotomy, providing a valuable source of numerical evidence in the field.

Gromov proved **nonsqueezing** in [Gro85], the groundbreaking result that a $2n$-ball of radius r in $\mathbb{C}^n$ can only embed symplectically into a cylinder $\mathbb{D}^2(R) \times \mathbb{C}^{n-1}$ if $r \leq R$. Nonsqueezing is rigid, as the ball has finite volume while the cylinder's volume is infinite. Conversely, many embedding problems are flexible: [Bir97], [MS12]. Characterizing embeddings is very difficult even for some of the simplest symplectic manifolds [McD09, McD11]. I study embeddings of toric domains, which are a large yet still tractable class of symplectic four-manifolds. They are also highly algebraic.

Definition 2.1. For Ω a region in the first quadrant of $\mathbb{R}^2$, the **toric domain** (X_Ω, ω_0) is

$$X_\Omega := \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \left(\pi|z_1|^2, \pi|z_2|^2\right) \in \Omega \right\}, \quad \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

If Ω is convex, then X_Ω is **convex**. If Ω is a polygon whose sides have rational slopes, X_Ω is **finite type**. We call Ω the **moment image** of X_Ω. See Figure 1 for key examples.



(a) **Ball** $B(a)$ (b) **Ellipsoid** $E(a,b)$ (c) **Polydisk** $P(a,b)$ (d) X_b

Figure 1: Moment images Ω, labeled by their toric domain X_Ω. All are convex.

Because $X_\Omega \overset{s}{\hookrightarrow} X_{c\Omega'}$ if $c >> 0$, we are interested in the most volume-filling and simplest nontrivial embeddings, captured by the **ellipsoid embedding function**

$$c_{X_\Omega}(a) := \inf \left\{ c \;\middle|\; (E(a,1), \omega_0) \overset{s}{\hookrightarrow} (X_{c\Omega}, \omega_0) \right\}.$$

McDuff and Schlenk in [MS12] found $c_{B(1)}$ has a surprising structure called an **infinite staircase**: $c_{B(1)}$ is either smooth or piecewise linear, with infinitely many nonsmooth points accumulating to a finite limit. Further researchers studied countable families of c_{X_Ω}, identifying and ruling out infinite staircases: [CFS17, Ush19, CHMP20, Cri]. But we still do not know why they appear.

Project 2.2. *Infinite staircases are both plentiful and highly non-generic. Can we characterize manifolds with infinite staircases?*

The unifying **toric staircases conjecture** from [CHMP20] is an entry point to Project 2.2. It asserts that (up to scale) there are exactly twelve convex polygons Ω with corners in $\mathbb{Q}^2$ and c_{X_Ω} having an infinite staircase. These twelve X_Ω all have circle actions on their boundary, and when reduced by this action they are symplectomorphic to an embedded submanifold of some complex projective space (making the reduced manifolds **Fano**). All evidence suggests the toric staircases conjecture is true. However, it is false if the "corners in $\mathbb{Q}^2$" hypothesis is removed. In fact, uncountably many X have staircases.

Theorem 2.3 ([MMW22, MPW23]). *The CGHMP conjecture holds for the X_b.*

- *In addition to $b = 1/3$, identified in [CHMP20], there is a bi-infinite family of Cantor sets of quadratic irrational b for which c_{X_b} has an infinite staircase.*
- *Membership in this family is controlled by the continued fraction of the solution to a simple quadratic equation in b. New members are generated from old following a Stern-Brocot tree.*

Theorem 2.3 suggests it is the number theory of the sizes of the tori making up X_Ω which truly controls staircases. It is the first classification of c_X for a continuous family of X. Work in progress with Holm, Magill, and Pires will extend to polydisks $P(1, b)$, with $b = 1/3$ replaced by $b = 1$.

Cristofaro-Gardiner, Magill, and McDuff are working to prove that if $\partial\Omega$ ever has irrational slope, c_{X_Ω} has no infinite staircase. I will build on their work towards an answer to Project 2.2:

Project 2.4. *If c_{X_Ω} has an infinite staircase, then X_Ω is finite type and $\partial\Omega$ has the same set of slopes as one of the twelve in the CGHMP conjecture.*

Project 2.4 is inspired by [BFMS21], in which the authors prove that a hyperbolic three-manifold containing infinitely many totally geodesic submanifolds must be arithmetic (i.e., controlled by number theory). One side of the analogy is that "steps" in an infinite staircase correspond to minimal surfaces in X_Ω; the other side is that Fano toric domains are highly controlled by the number theory of the lengths and slopes of the sides of $\partial\Omega$.

Beyond toric domains, there are other natural candidates for X whose symplectic forms come from number theory or algebraic geometry. Examples whose c_X I plan to compute include quotient symplectic forms on rational homology balls bounded by lens spaces, Hilbert modular varieties, and the space of flags on $\mathbb{C}^3$ (which is six-dimensional but highly symmetric), all of which generalize 4D toric domains in different ways.

2.1 Undergraduate research

I advised summer research in 2020 (under Jo Nelson) and 2022 (as lead organizer). In 2020, we considered polydisk embeddings, extending [Hut16a, Thm. 1.5]:

Theorem 2.5 ([DNN⁺22]). *If $1 \leq a \leq \frac{3}{2}, k \geq 3$, $P(a, 1) \overset{s}{\hookrightarrow} E\left(c\frac{2k+1}{2}, c\right) \Leftrightarrow P(a, 1) \subset E\left(c\frac{2k+1}{2}, c\right)$.*

Our proof used obstructions derived from the genus of minimal surfaces in the cobordism induced by the embedding. In 2022, we studied ellipsoid embedding functions with polydisk targets.

Theorem 2.6 ([FHM⁺22]). *Setting $\beta = \frac{6+5\sqrt{30}}{12}$, the function $c_{P(\beta,1)}$ has an infinite staircase.*

Usher found countably many $c_{P(\beta,1)}$ with an infinite staircase [Ush19], each with a counterpart staircase c_{X_b} [BHM⁺21, MM21]. Theorem 2.6 was the main evidence for my work in progress with Magill and Pires on proving Theorem 2.3 for the polydisk.

The proof of Theorem 2.3 does not compute the entire function c_{H_b}, which is particularly mysterious past the accumulation point. Future work with undergraduates will compute c_{X_Ω} when Ω is a pentagon or hexagon, which will provide evidence for Project 2.4. I also hope to optimize the algorithms for genus bounds used in [DNN⁺22].

3 Research program 2: Contact invariants and 3D topology

A **contact form** is a smooth one-form λ on an odd-dimensional manifold whose kernel hyperplane field is nonintegrable. Its **volume form** is $\lambda \wedge (d\lambda)^{\wedge n}$. An essential feature of a contact form is its **Reeb vector field** R, the unique vector field in the kernel of $d\lambda$ with $\lambda(R) = 1$. Its flow preserves λ, and its closed orbits are called **Reeb orbits**.

Every three-manifold Y carries many contact forms. Their Reeb dynamics detect the topology of Y. For example, every Reeb vector field on a three-manifold has a closed orbit ([Tau07]), and having exactly two Reeb orbits guarantees that Y is a lens space [HT09, CGHHL21].

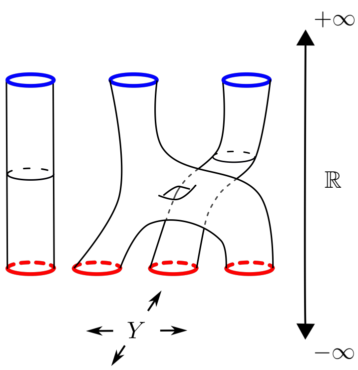


Figure 2: A 3D sketch of J-holomorphic curves defining the ECH differential.

Hutchings' **embedded contact homology** ("ECH") is a Floer-type homology theory for a contact three-manifold (Y, λ). ECH is generated by certain finite sets of Reeb orbits, and its differential is defined by counting "J-holomorphic" (or "pseudoholomorphic") curves in $\mathbb{R}\times Y$ asymptotic to cylinders over those orbits (Fig. 2). **J-holomorphic curves** are solutions to a nonlinear Cauchy-Riemann equation on $T(\mathbb{R}\times Y)$. If X is a cobordism from Y_+ to Y_-, it induces a cobordism map on their homologies.[1] ECH is invariant of λ and isomorphic to **Seiberg-Witten Floer homology** ("HM") and **Heegaard Floer homology** ("HF") [Tau10, KLT, CGH11].

HM and HF have significant strengths. ECH cobordism maps are defined via HM, while HF is more easily computable. The tripartite nature of the invariant is a feature, not a bug. Progress, such as Taubes' proof of the Weinstein conjecture, relies on porting tools and intuition from one version to another.

In §3.1 I explain my computations of ECH in varying topological settings. In §3.2 I explain my work on ECH capacities. §3.3 is dedicated to the ECH knot filtration and bounding symplectic knot cobordism volume. Finally, §3.4 introduces future work on computing ECH capacities within HF and characterizing L-spaces via ECH capacities.

3.1 Computing ECH

By far the most is known about λ which are invariant under a T^2 action. Much less is known when λ is only S^1-invariant. The simplest such class of examples are **prequantization bundles**, which are circle bundles over a surface with a contact form whose Reeb flow equals the fiber circle action.

Theorem 3.1 ([NW20]). *Let (Y, λ) be a prequantization bundle. Then $ECH_*(Y, \ker\lambda)$ is the exterior algebra of the homology of the surface at the level of chain complexes.*

Theorem 3.1 first appeared in a $\mathbb{Z}_2$-graded form in the unpublished thesis of Farris [Far11]; Nelson and I completed the J-holomorphic curve analysis and proved the $\mathbb{Z}$-graded version, using HM. Ferreira-Ramos used Theorem 3.1 to compute the "ECH capacities" (see §3.2) of the disk cotangent bundles of S^2 and $\mathbb{R}P^2$ [FR21], and Chen used it to compute further examples [Che23].

Prequantization orbibundles are precisely what they sound like – prequantization bundles over orbifolds. Some arise from "open book decompositions" by [CH13]. An **open book decomposition** is a fibration over S^1 of a three-manifold minus a link (the **binding**) by Seifert surfaces

[1] ECH cobordisms are backwards: $\partial X = Y_+ - Y_-$, and the map goes from $ECH(Y_+) \to ECH(Y_-)$.

(the **pages**). A contact form is **adapted** to an open book if its Reeb flow is transverse to the pages and tangent to the binding. A prequantization orbibundle arises when the Reeb flow is periodic.

Theorem 3.2 ([NW23a, NW23b])**.** *The ECH chain complex of the prequantization orbibundle λ adapted to an open book decomposition of S^3 with binding $T(p,q)$, $p,q>0$, is the exterior algebra of an orbifold Morse chain complex on the base when $p>2$, and when $p=2$ can have no differential.*

Applications of Theorem 3.2 to surface dynamics appear in §4.

3.2 ECH capacities

One of the most useful parts of the ECH package is its ability to capture length homologically.

Definition 3.3 ([Hut11])**.** The **ECH spectrum** of (Y,λ) is

$$0 = c_0(Y,\lambda) < c_1(Y,\lambda) \le c_2(Y,\lambda) \le \cdots \le \infty,$$

where $c_k(Y,\lambda)$ is the smallest L needed to represent a certain homology class of grading $2k$ using sets of Reeb orbits of length at most L. The **ECH capacities** of a symplectic filling (X,ω) of (Y,λ) are $c_k(X,\omega) = c_k(Y,\lambda)$.

ECH capacities obstruct symplectic embeddings,

$$\varphi : X \overset{s}{\hookrightarrow} X' \iff c_k(X,\omega) \le c_k(X',\omega') \text{ for all } k, \tag{3.1}$$

and more generally the ECH spectrum obstructs symplectic cobordisms. The idea is that $c_k(X',\omega') - c_k(X,\omega)$ is at least the area of a J-holomorphic curve in the cobordism $X' \setminus \varphi(X)$, which must exist by properties of the ECH cobordism map [HT13].

The ECH capacities of toric domains are algorithmic. My work in §2 in part uses computer programs to compute them up to $k = 25,000$. But identifying the curves counted by the ECH cobordism map or "U-map" (yet another count of curves, determining the specific grading $2k$ homology class defining c_k) requires passing through extremely complicated work of Taubes [Tau02]. They should be readily visible in the moment image of a toric domain as "tropical curves," a combinatorial projection of the true J-holomorphic curves.

Project 3.4. *Recently, McDuff-Siegel have found the curves underlying ECH capacities when Ω is one of CGHMP's Fano domains [MS23]. Extend their work to build a tropical ECH chain complex of a toric domain, including its differential, U-map, and cobordism maps.*

Finding evidence for Project 3.4 in the Hirzebruch surfaces appearing in Theorem 2.3 is work in progress with Holm, Magill, and Pires.

With Vinicius Ramos, I am completing the foundations of the combinatorial ECH chain complex of toric domains [RW], using [Tau02, Cho16]. Because the curves are still not explicit in [RW], Project 3.4 will provide a useful counterpoint. I hope it will serve as a motivating overarching program for a graduate student, because it combines tools from several fields in a new way.

Project 2.4 is also related to ECH capacities. The limit of $c_k(X)^2/(4k)$ is the symplectic volume of X by [CHR15]. My computational work towards Theorem 2.3 provides strong evidence that infinite staircases can only occur when the subleading asymptotics $c_k(X) - \sqrt{4k\,\mathrm{vol}(X)}$ do not converge, which includes the case of X_Ω finite type.

Finally, my work with Ramos will make it easy to compute ECH "barcodes." Barcodes, derived from topological data analysis, encode the lengths of Reeb orbits in the ECH chain complex. They are extremely useful in symplectic and contact geometry (see [PRSZ20]). I want to run an REU soon with the goal of writing and publishing code to compute ECH barcodes for toric domains.

3.3 Knots in ECH

One of the strengths of ECH is that it realizes HM/HF/ECH generators as multisets of embedded circles, which have not only a length but a link type.

My 2019 thesis investigated **knot-filtered ECH**, a spectral invariant measuring the linking of ECH generators with a fixed Reeb orbit. It was introduced by Hutchings in [Hut16b] when $H_1(Y) = 0$. If B is a Reeb orbit and $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a parameter depending on the linearized Reeb flow near B, $ECH^\ell(Y, \ker\lambda, B, \theta)$ is the homology of the subcomplex generated by orbit sets of linking number at most ℓ with B, modified by θ if B appears in the orbit set. I proved:

Theorem 3.5 ([Wei21])**.** *If $b_1(Y) = 0$, $ECH^\ell(Y, \ker\lambda, B, \theta)$ is defined and independent of λ, J.*

I applied Theorem 3.5 to lens spaces $Y = L(p, p-1)$ with B a component of the image of the Hopf link under the quotient $L(p, p-1) = S^3/\mathbb{Z}_p$, computed their knot-filtered ECH, and proved:

Theorem 3.6 ([Wei21])**.** *Contact forms on $L(p, p-1)$ adapted to open books with annulus pages and with certain irrational binding "rotation numbers" must have a third Reeb orbit γ. The length of γ is bounded from above by a function of its linking numbers with the binding components.*

Bechara Senior–Hryniewicz–Salomão have proved a version of Theorem 3.6 for more general (Y, λ) [BHS21] under a genericity hypothesis on λ.

With Nelson, I extended the definition and invariance of the knot filtration to some contact forms with $\theta \in \mathbb{Q}$, including those in Theorem 3.2: see [NW23a]. A formula for the knot filtration value of combinatorial ECH generators for toric contact forms will appear in [RW].

The knot filtration also refines the c_k to obstruct cobordisms. Denote by $c_{k,\ell}(Y, \lambda, B)$ the shortest length of a Reeb orbit set representing the grading $2k$ homology class (which is unique for S^3) with knot filtration at most ℓ (fixing λ fixes θ). If X is a symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) and $\Sigma \subset X$ is a symplectic cobordism from B_+ to B_-, we call (X, Σ) a **relative symplectic cobordism** from (Y_+, λ_+, B_+) to (Y_-, λ_-, B_-). With Roy and Yao, I am proving:

Theorem (in progress) 3.7 ([RWY])**.**

- *If (X, Σ) is a relative symplectic cobordism from (Y_+, λ_+, B_+) to (Y_-, λ_-, B_-), then $c_{k,\ell}(Y_+, \lambda_+, B_+) \geq c_{k,\ell}(Y_-, \lambda_-, B_-)$.*
- *The smallest a with $\operatorname{int}(P(12/5, 1)) \overset{s}{\hookrightarrow} B(a)$ is $a = 16/5$ [Hut16a], but if the complement of the embedding is a relative symplectic cobordism with $B_+ = B(a) \cap (\mathbb{C} \times \{0\})$ and $B_- = P(12/5, 1) \cap (\mathbb{C} \times \{0\})$ (both are Reeb orbits for $\lambda = \frac{1}{2}\sum_{i=1,2} x_i\,dy_i - y_i\,dx_i$), then $a \geq 17/5$.*

Etnyre and Golla have proved that $T(p, q)$ and $T(p', q')$ are cobordant in a weaker sense if $p \geq p'$ and $q \geq q'$ [EG22]; we will investigate whether this condition characterizes relative symplectic cobordism as well, and moreover identify the least possible volume for such cobordisms. I expect this work to inspire many projects for undergraduate summer research.

3.4 Directions for future work

The following projects reflect my overarching goal of interpreting features of HF and HM in ECH and vice versa. They will provide motivation for new research themes over the next several years.

3.4.1 HF capacities

Colin, Ghiggini, and Honda proved ECH and HF are isomorphic using open book decompositions, pioneering a method for computing ECH using the topological data of an open book [CGH10]. In [CHM$^+$21], Cristofaro-Gardiner–Humiliére–Mak–Seyfaddini–Smith defined the "link spectral invariant" in a version of HF for a mapping torus of a map isotopic to the identity. Chen proved that the link spectral invariant is quasimorphic to a variant of the ECH spectrum called the "PFH spectral invariants" [Che22]. Open books are mapping tori away from the binding, motivating:

Project 3.8. *Define a spectral invariant in HF using open book decompositions (it is well-known how to compute standard HF from an open book), and prove it is quasimorphic to the ECH spectrum.*

Project 3.8 will provide ongoing motivation over the next several years. Understanding the case of the positive torus knots discussed in §3.1 is the first place I will start, after which I will generalize to other open books.

3.4.2 L-spaces and ECH capacities

A rational homology three-sphere Y is an **L-space** if a reduced version of its HF has the same rank as its H_1. The "L-space conjecture" posits that being an L-space is equivalent to a topological condition – not carrying a taut foliation – and an algebraic one – having a non-left-orderable fundamental group [BGW13]. The same reduction appearing in HF can be performed using J-holomorphic curves in ECH, so it's natural to ask if being an L-space is equivalent to some Reeb dynamical condition.

This seems very hard to approach, as ECH is most useful when either the topology or Reeb dynamics is very simple, and non-lens-space L-spaces are far more complicated topologically and dynamically than any manifold whose ECH is well-understood. But there is hope in the work of Lin-Lipnowski on computing a key part of HM using a lower bound on the first eigenvalue of the Hodge-Laplacian [LL22] on a hyperbolic rational homology sphere. The ECH spectrum echoes the Laplace spectrum of a hyperbolic surface, exemplified in the volume property $\lim_{k\to\infty} c_k(Y,\lambda)^2/(2k) = \mathrm{vol}(Y,\lambda)$ proved in [CHR15]. In analogy to Lin-Lipnowski's work, I ask:

Project 3.9. *If Y carries a contact form λ that is "dynamically convex" and satisfies the lower bound $c_1(Y,\lambda)^2/\,\mathrm{vol}(Y,\lambda) > 2$, must Y be an L-space?*

Dynamical convexity is a technical condition we must assume by [ABHSa18], analogous to hyperbolicity. The idea is that L-spaces either have two Reeb orbits and are lens spaces (see [CDR23]), or infinitely many Reeb orbits which are cancelled in ECH by many differentials. If c_1 is large, either there are few short Reeb orbits (first scenario), or many J-holomorphic curves to cancel all short orbits (second scenario). Thus both scenarios give us hope that Y has very simple ECH, meaning Y is an L-space.

4 Research program 3: 2D dynamics via contact geometry

In my thesis [Wei21, Wei23] I used Theorem 3.6 to prove a quantitative version of the historic Poincaré-Birkhoff theorem that area-preserving diffeomorphisms of the annulus which rotate the boundaries in opposite directions have fixed points.[2]

Corollary 4.1 ([Wei21, Wei23])**.** *Let ψ be an area-preserving diffeomorphism of the annulus A which rotates a collar near both boundaries by $2\pi y_+$ (both measured clockwise). If $y_+ \neq \mathcal{V}(\psi)$, where $\mathcal{V}(\psi)$ is the "Calabi invariant" of ψ, then ψ has a periodic orbit γ.*

The utility of Corollary 4.1 is that its hypotheses only require computing y_+ and $\mathcal{V}(\psi)$.

Corollary 4.1 follows from Theorem 4.2 below, which requires some setup. The **action function** of ψ is $f : A \to \mathbb{R}$ with $df = \psi^*\beta - \beta$, where β is a primitive for the area form, and f along the outer boundary equals y_+. The action function measures local twisting via area: see Figure 3 for the case of a fixed point. The **mean action** of a periodic orbit γ of ψ is its average value on γ.

The action function has a global average, the **Calabi invariant**,

$$\mathcal{V}(\psi) = \frac{\int_A f\omega}{\int_A \omega},$$

a homomorphism to the reals which is usually only defined for Hamiltonian symplectomorphisms; the extension appearing here is due to Hutchings [Hut16b] in the case of the disk.

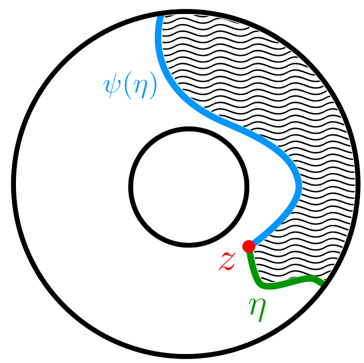


Figure 3: When $\psi(z) = z$, $f(z)$ equals the shaded area.

Theorem 4.2 ([Wei21, Wei23])**.** *Given ψ as in the first sentence of Corollary 4.1, and in addition $\mathcal{V}(\psi) < y_+$, then ψ has a periodic orbit whose mean action is less than $\mathcal{V}(\psi)$.*

Theorem 4.2 follows from Theorem 3.6 using open book decompositions of lens spaces with annulus pages. A contact form λ adapted to an open book induces a diffeomorphism of the page called the **return map**, which sends a point to its image under the Reeb flow for the shortest amount of time that point takes to return to the page. We can reverse this setup and construct λ with ψ as its return map. The diffeomorphism invariance of ECH allows us to estimate the quantities relevant to Theorem 4.2 using a simplified version of λ, as the diffeomorphism type of the resulting manifold depends only on the isotopy class and boundary rotation of ψ.

Theorem 3.6 was proved by combining computations using two contact forms on the boundaries of two different ellipsoids. Jonathan Trejos, a Ph.D. student at IMPA, is currently working out the ECH chain complex for "toric lens spaces," which will enable the following project:

Project 4.3. *(With graduate students.) Weaken the hypothesis on boundary rotation in Theorem 4.2.*

With Nelson, I am extending [Wei21, Wei23] to other surfaces. (See [PP22] for a similar result under a genericity assumption.)

[2] [Wei23] is a corrigendum to [Wei21] and is under review; results attributed solely to [Wei21] are not affected by the correction. Feel free to contact me with any questions.

Theorem 4.4 ([NW23b]). *Let ψ be an area-preserving diffeomorphism of a genus $(p-1)(q-1)/2$ surface with one boundary component in a certain isotopy class depending on p and q, whose action function is positive, and which is rotation by $\approx 2\pi/pq$ near the boundary. Then ψ has a periodic orbit whose mean action is less than $\sqrt{\mathcal{V}(\psi)/pq}$.*

The restriction on the isotopy class of ψ arises from requiring a contact form to which we can apply Theorem 3.2. Further computations of ECH will allow us to consider a wider variety of surfaces and isotopy classes of diffeomorphisms.

This work has influenced Project 3.8, which can be interpreted as a reverse project, that is, computing ECH spectral invariants from the return map, rather than using ECH spectral invariants to study the return map as we do here.

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