Infinite staircases of symplectic embeddings of ellipsoids into Hirzebruch surfaces

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Gromov nonsqueezing

Let $\omega = \sum_{i=1}^{2} dx_i \wedge dy_i$ be the std. symplectic form on $\mathbb{R}^4 = \mathbb{C}^2$.

Let $X, X' \subset \mathbb{R}^4$. A **symplectic embedding** $\varphi : X \overset{s}{\hookrightarrow} X'$ is a smooth embedding with $\varphi^*\omega = \omega$.

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Define the **ball**

$$B(c) := \left\{(z_1, z_2) \in \mathbb{C}^2 \middle| \pi|z_1|^2 + \pi|z_2|^2 \leq c\right\}$$

and the **cylinder**

$$Z(C) := \left\{(z_1, z_2) \in \mathbb{C}^2 \middle| \pi|z_1|^2 \leq C\right\}$$

Theorem (Gromov '84)

$B(c) \overset{s}{\hookrightarrow} Z(C) \Rightarrow c \leq C$ *(notice: no volume obstruction!).*

The McDuff-Schlenk Fibonacci stairs

Generalize the ball to the **ellipsoid**

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \middle| \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}$$

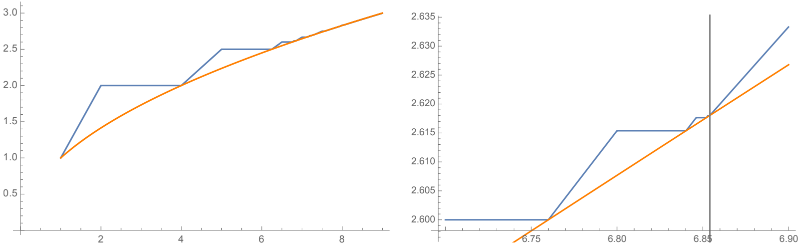
Define the **ellipsoid embedding function** of the ball

$$c_0(a) := \inf \left\{ \mu > 0 \middle| E(1,a) \overset{s}{\hookrightarrow} B(\mu) \right\}$$

We know $c_0(a) \geq \sqrt{a}$ from the volume obstruction.

Theorem (McDuff-Schlenk '12)

c_0 is piecewise linear or smooth and nonsmooth at infinitely many points. A subsequence of nonsmooth points accumulates from below at (τ^4, τ^2), where $\tau = \frac{1+\sqrt{5}}{2}$. For large a, $c_0(a) = \sqrt{a}$.



(a) Plot of c_0.

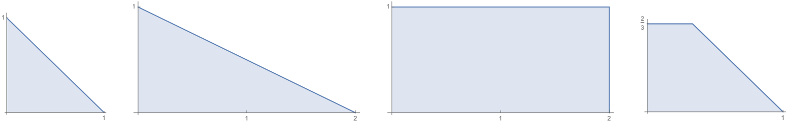
(b) Zoomed. Gray: $a = \tau^4 = \left(\frac{1+\sqrt{5}}{2}\right)^4$.

Figure: Orange: volume obstruction $\sqrt{a}$. Blue: plot of c_0.

- The steps ascend from below.
- The x-coordinates of the outer corners are $2, 5, \frac{13}{2}, \frac{34}{5}, \frac{89}{13}, \ldots$

Toric manifolds and toric domains

A **toric domain** X_Ω in $\mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}^2_{\geq 0}$ under the map $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$.



(a) $B(1)$ (b) $E(1,2)$ (c) $P(1,2)$ (d) $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2_{\frac{1}{3}}$

Let M_Ω be the symplectic toric manifold with one of the above polytopes. Using Cristofaro-Gardiner–Holm–Mandini–Pires '20:

$$E(a,b) \overset{s}{\hookrightarrow} X_\Omega \Leftrightarrow E(a,b) \overset{s}{\hookrightarrow} M_\Omega$$

Other ellipsoid embedding functions

Let $\mu X_\Omega = X_{\mu\Omega}$ (i.e. $|z_i|^2$ scales by μ.)

Define the **ellipsoid embedding function** of X_Ω by

$$c_{X_\Omega}(a) := \inf\left\{\mu > 0 \middle| E(1,a) \stackrel{s}{\hookrightarrow} \mu X_\Omega\right\} \geq \sqrt{\frac{a}{\mathrm{vol}(X_\Omega)}}$$

For a large enough $c_{X_\Omega}(a) = \sqrt{\frac{a}{\mathrm{vol}(X_\Omega)}}$.

We say $c_{X_\Omega}(a)$ has an **infinite staircase** if it is nonsmooth at infinitely many points.

What is known

Based on the vertices and edges of Ω, we know:

Ω has integer vertices: The most is known.

Cristofaro-Gardiner–Holm–Mandini–Pires '20 find 12 Ωs with infinite staircases, all ascending, including $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2_{\frac{1}{3}}$. They conjecture there are no others.

Ω has rational edge slopes, irrational vertices: One result to date.

Usher '18 found infinitely many ascending infinite staircases for polydisks $P(1, b)$, $b \in \mathbb{R} - \mathbb{Q}$.

Ω has irrational edge slopes: Nothing is known.

New infinite staircases

Let Ω_b be the Delzant polytope of the Hirzebruch sfc. $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2_b$, i.e., the trapezoid with corner $(b, 1-b)$. Let $c_b := c_{X_{\Omega_b}}$.

Theorem (Bertozzi-Holm-Maw-McDuff-Mwakyoma-Pires-W i.p.)

Let

$$b_0 = \frac{5(165 - 7\sqrt{5})}{2698} \approx 0.2767745073$$

c_{b_0} *has an infinite staircase*
whose steps **descend** *to accumulate at*

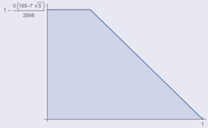


Figure: Ω_{b_0}

$$\left(\frac{2443 + 3\sqrt{5}}{418}, \frac{\sqrt{281981 - 2124\sqrt{5}}}{209} \right) \approx (5.86054594, 2.51927208)$$

c_{b_0} near the accumulation point

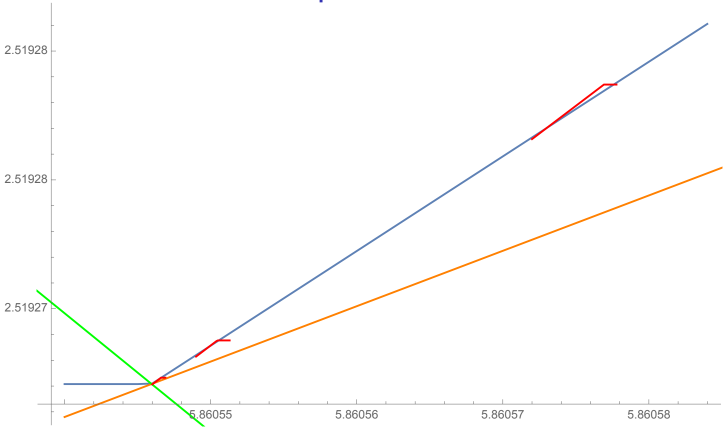
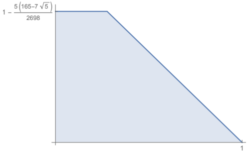


Figure: Max of blue and the many reds is c_{b_0}. Orange: volume constraint. Green: crosses c_{b_0} at the accumulation point. The stairs descend instead of ascending like c_0's.

Accumulation points

Recall Ω_b is the convex hull of $(0,1-b),(b,1-b),(1,0),(0,0)$.



Theorem (C-G–H–M–P '20)

If c_b has an infinite staircase, the x-coordinate of its accumulation point, denoted by acc(b), is the larger of the solutions to

$$x^2-\left(\frac{(3-b)^2}{1-b^2}-2\right)x+1=0$$

The accumulation point will always be on the volume obstruction.

The accumulation point curve

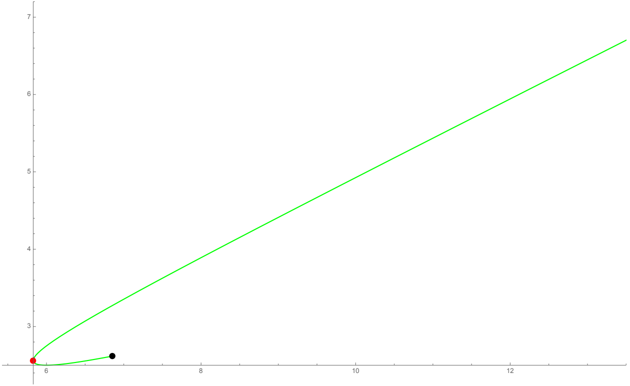


Figure: Green: the parameterized curve $\left(\text{acc}(b), \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}}\right)$. Black: the accumulation point (τ^4, τ^2) of the Fibonacci stairs. Red: accumulation point of the $b = \frac{1}{3}$ stairs.

Theorem (B–H–M^3–P–W in various states of progress)

There are five infinite sequences of bs where c_b has an ∞ staircase:

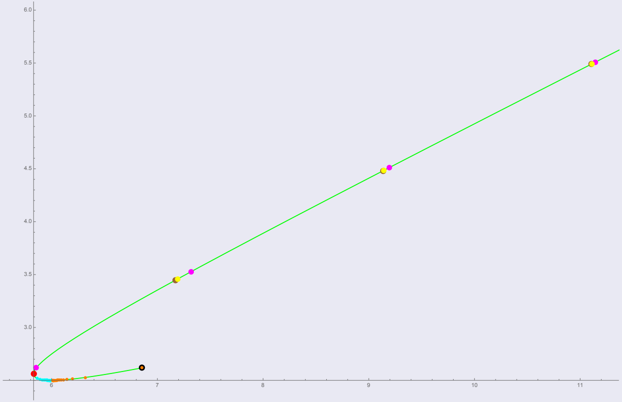
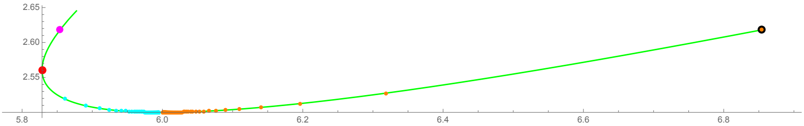


Figure: *Orange, pink, and yellow are ascending staircases (x-values of nonsmooth points increase). Cyan and brown are descending.*

There are likely many more such sequences of infinite staircases.

Zooming in near $b = \frac{1}{5}$



The accumulation point of c_{b_0} is the leftmost cyan point.

The minimum of $\sqrt{\frac{\mathrm{acc}(b)}{\mathrm{vol}(X_{\Omega_b})}}$ occurs at $b = \frac{1}{5}$.

$c_{\frac{1}{5}}$ likely does not have an infinite staircase:

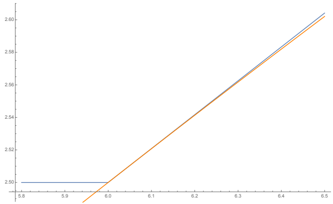


Figure: $c_{\frac{1}{5}}$

ECH capacities and ellipsoid embedding functions

X_Ω has **ECH capacities** $0 = c_0(X) < c_1(X) \leq c_2(X) \cdots \leq \infty$.

If Ω is convex, $c_{X_\Omega}(a) = \sup_k \left\{ \frac{c_k(E(1,a))}{c_k(X_\Omega)} \right\}$ (Cristofaro-Gardiner '19).

This is handy, because $c_k(X_\Omega)$ is combinatorial.

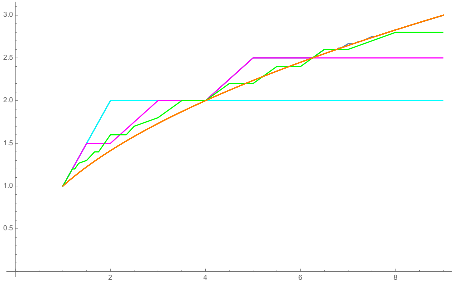


Figure: Orange: volume obstruction. Blue: c_0. The obstructions $\frac{c_2(E(1,a))}{c_2(B(1))}$, $\frac{c_5(E(1,a))}{c_5(B(1))}$, $\frac{c_{20}(E(1,a))}{c_{20}(B(1))}$.

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But c_b is still a supremum over an infinite set!

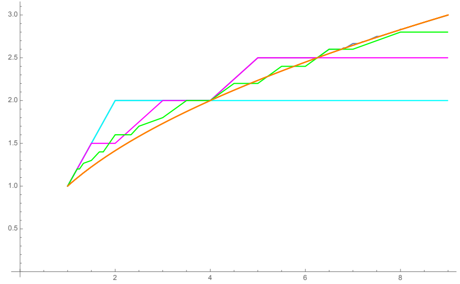
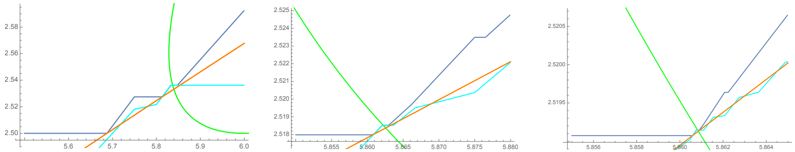


Figure: Orange: volume obstruction. Blue: c_0. The obstructions $\frac{c_2(E(1,a))}{c_2(B(1))}$, $\frac{c_5(E(1,a))}{c_5(B(1))}$, $\frac{c_{20}(E(1,a))}{c_{20}(B(1))}$.

Identifying obstructive capacities

We ruled out many b for which

$$c_b(\mathrm{acc}(b)) \geq \max_{k=1,\ldots,25{,}000} \left\{ \frac{c_k(E(1,\mathrm{acc}(b)))}{c_k(X_{\Omega_b})} \right\} > \sqrt{\frac{\mathrm{acc}(b)}{\mathrm{vol}(X_{\Omega_b})}}$$



(a) $c_{0.3}, k = 125$ (b) $c_{0.275}, k = 2564$ (c) $c_{0.2765}, k = 18,559$

Figure: Orange: volume obstruction. Blue: c_b. Green: accumulation point curve. Cyan: $\frac{c_k(E(1,a))}{c_k(X_{\Omega_b})}$.

Unviable regions of b

Each capacity rules out an infinite staircase for an interval of bs.

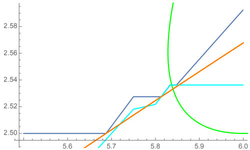


Figure: $c_{0.3}, k = 125$

For example, $\frac{c_{125}(E(1,\text{acc}(b)))}{c_{125}(X_{\Omega_b})} > \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}}$ for at least

$$0.277 < b < 0.32475$$

And $\frac{c_{2564}(E(1,\text{acc}(b)))}{c_{2564}(X_{\Omega_b})} > \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}}$ for at least

$$0.274398 < b < 0.27643$$

Periodic continued fractions: hidden structure of the steps

Now we've ruled out many bs, we look for staircases in what's left.

The **continued fraction expansion** of a number a is the sequence $[a_0, a_1, a_2, \dots]$ where $a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$.

In known ∞ staircases, x-coords of the outer corners of stairs have periodic CFs. E.g. the Fibonacci stairs have outer corners

$$\frac{13}{2} = [6,2], \frac{34}{5} = [6,1,4], \frac{89}{13} = [6,1,5,2], \frac{233}{34} = [6,1,5,1,4], \dots$$

i.e. $[6, \{1,5\}^k, 2]$ or $[6, \{1,5\}^k, 1, 4]$

Accumulation points have infinite periodic CFs: $\tau^4 = [6, \{1,5\}^\infty]$.

Climb (or descend) the periodic CFs to infinity!

We looked for sequences a_k such that:

- a_k has a periodic continued fraction
- $a_k \to \text{acc}(b)$ and $\max_{k=1,\ldots,25{,}000}\left\{\frac{c_k(E(1,\text{acc}(b)))}{c_k(X_{\Omega_b})}\right\} < \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}}$

Such a_k could be outer corners of stairs in ∞ staircases. It worked!

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$0 \leq b < \frac{1}{5}$: $[6, \{1+2n, 5+2n\}^k, End_i(n)]$, where
$End_1(n) = 2+2n$, $End_2(n) = \{1+2n, 4+2n\}$

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$\frac{1}{5} < b < \frac{1}{3}$: $[5, 1, 6+2n, \{5+2n, 1+2n\}^k, end_i(n)]$, where
$end_1(n) = 4+2n$, $end_2(n) = \{5+2n, 2+2n\}$
c_{b_0} is the $n=0$ case, $b_0 = \mathrm{acc}^{-1}([5, 1, 6, \{5, 1\}^\infty])$

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$\frac{1}{3} < b < 1$: yellow $[\{7+2n, 5+2n, 3+2n, 1+2n\}^k, 6+2n]$;
$[7+2n, \{5+2n, 1+2n\}^k, end_i(n)]$;
$[\{5+2n, 1+2n\}^k, end_i(n)]$

Proving we have a staircase

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Would take us too long for today!

Thanks!

Thank you!