Corrigendum to “Mean action of periodic orbits of area-preserving annulus diffeomorphisms”

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1 Introduction

This corrigendum corrects two mathematical errors in [2]. The main results are not affected, but certain sections (§3 and §5, and parts of §6) require modification, with the most significant changes in §5 and the beginning of §6.

1. The statement and proof of [2, Prop. 3.1] are incorrect; in particular, in Step 3 of the proof, the contact manifold is misidentified as $L(y_+ - y_- + F, y_+ - y_- + F - 1)$, when in fact it is $L(F, F-1)$. While the rest of the paper (barring the second error below) is entirely correct as far as we know, several sections (§5.2, §5.3, §6.1, and §6.2) require adjustments as they now only apply in the case when $y_+ - y_- + F = F$, i.e., when $y_+ = y_-$.

2. In the proof of [2, Prop. 6.3] we need to show that a Reeb orbit satisfying certain action and intersection number inequalities is nonempty. We accomplish this by showing that its action is positive in equation (6.16). However, the original argument is incorrect, as it relies on a function C of N, defined in (6.14), to be uniformly bounded below one as N goes to infinity. This is not the case.

   Our new method bypasses Lemma 6.2 and instead uses the more straightforward [1, Lem. 3.2]. The main new feature is that instead of using the Reeb dynamics of a single lens space to capture the dynamics of an annulus map, we use two different lens spaces.

We correct the first error in §2 and the second error in §3. Throughout this corrigendum we use the notation of [2].

1.1 Acknowledgements

We would like to heartily thank Abror Pirnapasov for pointing out the second error and for his comments on this corrigendum, as well as Tara Holm, Jo Nelson, and Michael Hutchings for helpful discussions.

2 Changes to §3, §5.2, §5.3, and §6.1

2.1 Correction to Proposition 3.1 and its proof

The correct version of [2, Propositon 3.1] is the following:

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Proposition 2.1. *Let ψ be an area-preserving diffeomorphism of (A,ω) which is rotation by $2\pi y_\pm$ near $\partial_\pm A$, whose flux is $F \in \mathbb{Z}$, for which both y_+ and $-y_- + F$ are irrational, and whose action function f is positive. Then there is a contact form $\lambda_{\tilde{\psi}}$ on $L(F, F-1)$ for which*

1. *An open book decomposition (B_F, P_F) of $L(F, F-1)$ with abstract open book (A, D_F) is adapted to $\lambda_{\tilde{\psi}}$. Let A_0 denote the closure of the zero page. The return time of the Reeb flow from $\mathring{A}_0$ to $\mathring{A}_0$ is given by the action function f, and ψ is the return map of $(\lambda_{\tilde{\psi}}, B_F, P_F)$.*

2. *The binding orbits have action one, are elliptic, and have rotation numbers $\frac{1}{y_+}$ and $\frac{1}{-y_-+F}$ in the trivializations which have linking number zero with their component of B_F with respect to A_0.*

3. *Let $\{|B_F|\}$ denote the set of components of B_F. There is a bijection $\mathcal{P}(\psi) \cup \{|B_F|\} \to \mathcal{P}(\lambda_{\tilde{\psi}})$. The symplectic action of the Reeb orbit corresponding to $\gamma \in \mathcal{P}(\psi)$ is $\mathcal{A}(\gamma)$, and its intersection number with the page A_0 is $\ell(\gamma)$.*

4. *The contact volume satisfies* $\mathrm{vol}(L(F, F-1), \lambda_{\tilde{\psi}}) = 2\mathcal{V}(\psi)$.

Proof. Step 1 holds without change, and Steps 4-5 can be replaced with exact analogues. Replace Steps 2-3 with the following:

Step 2: The closed manifold

Consider the oriented coordinates (ρ_+, μ_+, t_+) and (ρ_-, t_-, μ_-) on the solid tori $\mathbb{T}_\pm = \mathbb{D}^2(\epsilon_\pm) \times (\mathbb{R}/2\pi\mathbb{Z})$, where $\rho_\pm \in [0, \epsilon_\pm]$ and $\mu_\pm \in \mathbb{R}/2\pi\mathbb{Z}$ are coordinates on $\mathbb{D}^2(\epsilon_\pm)$ and the coordinate on $\mathbb{R}/2\pi\mathbb{Z}$ is $t_\pm \in \mathbb{R}/2\pi\mathbb{Z}$. Let $g_\pm : \mathring{M}_\psi \to \mathbb{T}_\pm$ be given by

$$
\begin{aligned}
g_+(x, y, \theta) &= \left(\sqrt{1-x}, 2\pi\theta, y + 2\pi\theta y_+\right) \\
g_-(x, \theta, y) &= \left(\sqrt{x+1}, y + 2\pi\theta(y_- - F), 2\pi\theta\right),
\end{aligned}
$$

in oriented coordinates on both the domain and target. Because $F \in \mathbb{Z}$, the map g_- is well-defined.

Let Y_ψ denote the union of $\mathring{M}_\psi$ with the $\mathbb{T}_\pm$s via the $g_\pm$s.

Step 3: Open book decomposition

Denote by B_F the subset of Y_ψ where $\{\rho_\pm = 0\}$. Let $P_F : Y_\psi - B_F \to S^1$ be given by $(t, z) \mapsto t$. The preimages $P_F^{-1}(t)$ are diffeomorphic to $\mathring{A}$. We claim that P_F is a projection map for an open book decomposition with page A.

The meridional direction near the component of B_F corresponding to $\partial_\pm A$ is given by $\partial_{\mu_\pm}$, which extends to $\mathring{M}_\psi$ as $-y_+\partial_y + \frac{1}{2\pi}\partial_\theta$ near $\partial_+ A$ and $(-y_- + F)\partial_y + \frac{1}{2\pi}\partial_\theta$ near $\partial_- A$. The direction ∂_θ is transverse to the fibers of P_F. Choose smooth monotone interpolations

- $\delta_+ : [-1, 1] \to [-y_+, 0]$ with $\delta_+|_{[-1, 1-\epsilon_+^2]} = 0$ and $\delta_+(1) = -y_+$,
- $\delta_- : [-1, 1] \to [0, -y_- + F]$ with $\delta_-|_{[\epsilon_-^2 - 1, 1]} = 0$ and $\delta_-(-1) = -y_- + F$.

Let V be the vector field

$$
V = (\delta_+(x) + \delta_-(x))\partial_y + \frac{1}{2\pi}\partial_\theta,
$$

which is transverse to the pages of P_F and equals $\partial_{\mu_\pm}$ near B_F.

We claim that the return map of the flow of V from $P_F^{-1}(0)$ to itself is homotopic (relative to ∂A) to the F-fold right-handed Dehn twist D_F. Because the coefficient of ∂_θ in V is $\frac{1}{2\pi}$, it takes at least time 2π to send $P^{-1}(0)$ to itself. The return map of the time 2π flow of V near the $\partial_+ A$ component of $P^{-1}(0)$ is

$$(x, y, 0) \mapsto (x, y - 2\pi y_+, 1) \sim (x, y, 0),$$

while near the $\partial_- A$ component, the return map is

$$(x, 0, y) \mapsto (x, 1, y + 2\pi(-y_- + F)) \sim (x, 0, y + 2\pi F),$$

where we do not make the simplification $y + 2\pi F \sim y \in \mathbb{R}/2\pi\mathbb{Z}$ to emphasize the F-fold right-handed Dehn twist. □

Throughout the paper, $\tilde{p}$ should be replaced with F; below, we discuss only changes to notation, results, and proofs, and leave it to the reader to make the necessary changes to the connecting text.

2.2 Corrections to §5.2

The correct version of [2, Lem. 5.5] is the following:

Lemma 2.2. *The rotation numbers of $e_\pm^F$ in the trivializations of $\ker \lambda_{\tilde{\psi}}$ which have linking number zero with $e_\pm^F$ with respect to their Seifert surfaces are $\frac{F}{y_+} - 1$ and $\frac{F}{-y_- + F} - 1$.*

Proof. Replace $\tilde{p}$ with F in the proof of [2, Lem. 5.5]. □

[2, Prop. 5.4] holds in its entirety only when $y_+ = y_-$. In general, we can only expect one of the rotation numbers of $e_\pm$ to agree with those of Propositon 2.1. The corrected version is as follows:

Proposition 2.3. *If $\frac{F}{y_+} - 1, \frac{F}{-y_- + F} - 1 \in \mathbb{R} - \mathbb{Q}$, there are nondegenerate contact forms $\lambda_\pm^F$ on $L(F, F-1)$ satisfying*

1. *$\ker \lambda_\pm^F$ and $\ker \lambda_{\tilde{\psi}}$ are contactomorphic.*
2. *Under the diffeomorphism of 1., the orbits $e_\pm$ of $\lambda_{\tilde{\psi}}$ are both also simple nondegenerate elliptic Reeb orbits for $\lambda_\pm^F$, and $\lambda_\pm^F$ have no other simple Reeb orbits.*
3. *(a) The nullhomologous cover e_+^F of e_+ has rotation number $\frac{F}{y_+} - 1$ and as a Reeb orbit of λ_+^F when computed in the trivialization of $\ker \lambda_+^F$ which has linking number zero with e_+^F with respect to its Seifert surface S_+.*
   (b) The nullhomologous cover e_-^F of e_- has rotation number $\frac{F}{-y_- + F} - 1$ as a Reeb orbit of λ_-^F when computed in the trivialization of $\ker \lambda_-^F$ which has linking number zero with e_-^F with respect to its Seifert surface S_-.

Proof. The proof is identical to that of [2, Prop. 5.4], except we define

$$\mathfrak{q}_F^* \lambda_\pm^F = \lambda_{(a_\pm, b_\pm)},$$

where

$$a_+ = F - y_+, b_+ = y_+, a_- = F - (-y_- + F) = y_-, b_- = F - y_-.$$

□

The connecting text in the rest of §5.2 can be read as-is, replacing $\tilde{p}$ with F and doubling each result or discussion to apply to both $\lambda_{\pm}^F$. We thus obtain:

Proposition 2.4. *1. The generators of $ECC_*(L(F,F-1),\lambda_{\pm}^F,J)$ correspond to points (d,m_+) in the second skew quadrant determined by the x-axis and the line $y=Fx$:*

$$(d,m_+)\leftrightarrow e_+^{m_+}e_-^{m_-},\ \textit{where}\ \frac{m_+-m_-}{F}=:d.$$

2. There is a bijection between generators and $2\mathbb{Z}_{\geq 0}$ given by, in the case of λ_+^F, the order in which a line of slope y_+ moving northwest passes through the points in the second skew quadrant in 1.; the bijection in the case of λ_-^F is given by a line of slope y_-.

2.3 Corrections to §5.3

Note that by simple geometry, the y-coordinate of the y-intercept of the line through (d,m_+) of slope $y_\pm$ equals $f_\pm\mathcal{F}_{e_\pm}(e^{m_+}e^{m_-})$, where $f_+=y_+$ and $f_-=-y_-+F$, the values of the action function on $\partial_\pm A$. This proves the correct version of [2, Prop. 5.9]:

Proposition 2.5.

$$ECH_{2k}^{\mathcal{F}_{e_+}\leq\ell}\left(L(F,F-1),\xi_{\tilde{\psi}},e_+,\frac{1}{y_+}-\frac{1}{F}\right)=\begin{cases}\mathbb{Z}/2\mathbb{Z} & \textit{if } \ell\geq N_{w_+^F(k)}\left(\frac{1}{y_+}-\frac{1}{F},\frac{1}{F}\right)\\ 0 & \textit{else}\end{cases}$$

$$ECH_{2k}^{\mathcal{F}_{e_-}\leq\ell}\left(L(F,F-1),\xi_{\tilde{\psi}},e_-,\frac{1}{-y_-+F}-\frac{1}{F}\right)=\begin{cases}\mathbb{Z}/2\mathbb{Z} & \textit{if } \ell\geq N_{w_-^F(k)}\left(\frac{1}{F},\frac{1}{-y_-+F}-\frac{1}{F}\right)\\ 0 & \textit{else}\end{cases}.$$

Here $w_\pm^F(k)$ are defined so that $N_{w_\pm^F(k)}(a,b)$ is the k^{th} largest of the sequence of nonnegative integer linear combinations m_+a+m_-b where m_+-m_- is divisible by F; notice that therefore $w_\pm^F$ also depends on a,b, but we omit this from the notation.

2.4 Corrections to §6.1

[2, Prop. 6.1] must be corrected to

Proposition 2.6. *Let λ be a contact form on $L(F,F-1)$ contactomorphic to the contact form λ_F from [2, Lem. 2.6]. Suppose that both binding components $b_\pm$ of the open book decomposition (H_F,Π_F) are elliptic for λ (in particular, with irrational rotation numbers). Then, for all $\epsilon>0$, for all sufficiently large integers k there is an orbit set α_k not including either $b_\pm$ and nonnegative integers $m_{k,\pm}$ for which*

$$I(b_+^{m_{k,+}}\alpha_k b_-^{m_{k,-}})=2k$$

$$\mathcal{A}(\alpha_k)\leq\sqrt{2k(\mathrm{vol}(L(F,F-1),\lambda)+\epsilon)}-m_{k,+}\mathcal{A}(b_+)-m_{k,-}\mathcal{A}(b_-)\tag{2.1}$$

$$\alpha_k\cdot A_0\geq N_{w_+^F(k)}\left(\mathrm{rot}(b_+),\frac{1}{F}\right)+N_{w_-^F(k)}\left(\frac{1}{F},\mathrm{rot}(b_-)\right)-m_{k,+}\,\mathrm{rot}(b_+)-m_{k,-}\,\mathrm{rot}(b_-).\tag{2.2}$$

Proof. The proof is identical to that of [2, Prop. 6.1], with the appropriate replacements. □

3 Corrections to §6.2

We require an additional lemma.

Lemma 3.1. *For infinitely many values of k, we have*

$$N_{w^F_\pm(k)}(a, 1/F) \geq N_k(a, 1).$$

Proof. If $N_{w^F_\pm(k)}(a, 1/F) = m_+a + m_-/F$ and m_- is divisible by F, then $m_+a + m_-/F$ appears in the sequence $N(a, 1)$, and is at least the k^{th} term since each term in the $N_{w^F_\pm}(a, 1/F)$ sequence with m_- decreased by a multiple of F appears as a term in the $N(a, 1)$ sequence, and there are exactly $k - 1$ of these. □

We omit [2, Lem. 6.2]. [2, Prop. 6.3] must be slightly corrected to the following:

Proposition 3.2. *Let ψ be an area-preserving diffeomorphism of (A, ω) which is rotation by $2\pi y_\pm$ near $\partial_\pm A$, whose flux applied to the class of the $(x, 0)$ curve is $F \in \mathbb{Z}$, whose action function f is positive, and where y_+ and $-y_- + F$ are irrational.*

Let $\mathcal{A}_N$ denote the total action computed with $f_{(\psi, y_+ + N, \beta)}$ rather than with $f_{(\psi, y_+, \beta)}$. If

$$\mathcal{V}(\tilde{\psi}) < \max\{y_+, -y_- + F\}$$

then for all integers $N \geq 2\max\{y_+, -y_- + F\} - 3\min\{y_+, -y_- + F\}$,

$$\inf\left\{\frac{\mathcal{A}_N(\gamma)}{\ell(\gamma)}\middle|\gamma \in \mathcal{P}(\psi)\right\} \leq \sqrt{\operatorname{hm}(y_+ + N, -y_- + F + N)(\mathcal{V}(\tilde{\psi}) + N)}. \tag{3.1}$$

Proof. Note that the hypotheses imply that also $\frac{F}{y_+} - 1$ and $\frac{F}{-y_- + F} - 1$ are irrational, so we can apply Proposition 2.6.

The proof is the same until we apply the conclusion of [2, Lem. 6.2] to show that (2.2) is a strict inequality; instead we use [1, Lem. 3.2], whose conclusion is that for some constant c,

$$N_k(a, b) \geq \sqrt{2abk - ck^{\frac{1}{2}}}. \tag{3.2}$$

(2.2) implies

$$\alpha_k \cdot A_0 \geq N_{w^F_+(k)}\left(\frac{1}{y_+} - \frac{1}{F}, \frac{1}{F}\right) + N_{w^F_-(k)}\left(\frac{1}{F}, \frac{1}{-y_- + F} - \frac{1}{F}\right) - m_{k,+}\left(\frac{1}{y_+} - \frac{1}{F}\right) - m_{k,-}\left(\frac{1}{-y_- + F} - \frac{1}{F}\right),$$

which is weakest when $m_{k,\pm}$ are at least as large as the coefficients of $\frac{1}{y_+} - \frac{1}{F}$ and $\frac{1}{-y_- + F} - \frac{1}{F}$ in their respective $N_{w^F_\pm(k)}$ terms. Therefore we assume this, which means to show (2.2) is strict it is enough to show

$$N_{w^F_+(k)}\left(\frac{1}{y_+}, \frac{1}{F}\right) + N_{w_-(k)}\left(\frac{1}{F}, \frac{1}{-y_- + F}\right) > \frac{m_{k,+}}{y_+} + \frac{m_{k,-}}{-y_- + F}.$$

By Lemma 3.1 (increasing k if necessary), (3.2) with k large enough, (2.1), Proposition 2.1 4., and setting $m = \min\{y_+, -y_- + F\}, M = \max\{y_+, -y_- + F\}$, it suffices to show

$$\sqrt{\frac{2k}{m}} + \sqrt{\frac{2k}{M}} > \frac{\sqrt{4k\mathcal{V}(\tilde{\psi})}}{m}.$$

Using our assumption that $\mathcal{V}(\tilde{\psi}) < M$ and simplifying, it is enough to show

$$\begin{aligned}\sqrt{\frac{1}{m}} + \sqrt{\frac{1}{M}} &> \frac{\sqrt{2M}}{m} \\ \frac{1}{m} + \frac{2}{\sqrt{mM}} + \frac{1}{M} &> \frac{2M}{m^2} \\ mM + 2m\sqrt{mM} + m^2 &> 2M^2.\end{aligned}$$

Computing m and M with $y_+ + N$ and replacing $\sqrt{mM}$ with m, it is enough to show

$$\begin{aligned}(m+N)(M+N) + 3(m+N)^2 &> 2(M+N)^2 \\ 4N^2 + (7m+M)N + 3m^2 + mM &> 2N^2 + 4MN + 2M^2 \\ 2N^2 + (7m-3M)N + (3m^2 + mM - 2M^2) &> 0,\end{aligned}$$

which follows if $N \geq 2M - 3m$. Therefore $\alpha_k \neq \emptyset$.

Combining (2.1) and (2.2) for the corresponding orbit set, reducing to a single orbit γ as at the start of the original proof of [2, Prop. 6.1], and using Proposition 2.1 4. gives us an upper bound on $\mathcal{A}_N(\gamma)/\ell(\gamma)$ of

$$\frac{\sqrt{2k(2(\mathcal{V}(\tilde{\psi})+N)+\epsilon)} - m_{k,+} - m_{k,-}}{\sqrt{2k\left(\frac{1}{y_++N} - \frac{1}{F+2N}\right)} + \sqrt{2k\left(\frac{1}{-y_-+F+N} - \frac{1}{F+2N}\right)} - m_{k,+}\left(\frac{1}{y_++N} - \frac{1}{F+2N}\right) - m_{k,-}\left(\frac{1}{-y_-+F+N} - \frac{1}{F+2N}\right)}.$$

By the same logic as in the original proof, this is maximized when $m_{k,\pm} = 0$, therefore by simplifying we have

$$\frac{\mathcal{A}_N(\gamma)}{\ell(\gamma)} \leq \frac{\sqrt{2(\mathcal{V}(\tilde{\psi})+N)}}{\sqrt{\frac{1}{y_++N} - \frac{1}{F+2N}} + \sqrt{\frac{1}{-y_-+F+N} - \frac{1}{F+2N}}}. \tag{3.3}$$

To show that the right hand side of (3.3) is at most the right hand side of (3.1), we must show:

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{\frac{1}{m+N} - \frac{1}{F+2N}} + \sqrt{\frac{1}{M+N} - \frac{1}{F+2N}}} &\leq \sqrt{\frac{2}{\frac{1}{m+N} + \frac{1}{M+N}}} \\ \sqrt{\left(\frac{1}{m+N} - \frac{1}{F+2N}\right)\left(\frac{1}{M+N} - \frac{1}{F+2N}\right)} &\geq \frac{1}{F+2N} \\ \frac{1}{(m+N)(M+N)} - \frac{1}{(m+N)(F+2N)} - \frac{1}{(M+N)(F+2N)} &\geq 0 \\ F + 2N - (M+N) - (m+N) &\geq 0.\end{aligned}$$

It remains to show we can always assume $F \geq M + m$. Notice

$$F \geq M + m \Leftrightarrow F \geq y_+ - y_- + F \Leftrightarrow y_- \geq y_+.$$

Now if $\psi(x,y) = (\psi^1(x,y), \psi^2(x,y))$, let $\widehat{\psi}(x,y) = (-\psi^1(-x,y), \psi^2(-x,y))$; it can be checked that $\widehat{\psi}^*\omega_0 = \psi^*\omega_0 = \omega_0$ and that the rotation number of $\widehat{\psi}$ along $\partial_\pm A$ is $y_\mp$. In particular, if $y_- < y_+$ for ψ, then the opposite is true for $\widehat{\psi}$.

From the construction of the action function, with $\beta = \frac{x}{2\pi}\, dy$, we have

$$
\begin{aligned}
2\pi df(x,y) &= \psi^1(x,y)\psi^2_x(x,y)\, dx + (\psi^1(x,y)\psi^2_y(x,y) - x)\, dy \\
2\pi d\widehat{f}(x,y) &= \psi^1(-x,y)\psi^2_x(-x,y)\, dx + (-\psi^1(-x,y)\psi^2_y(-x,y) - x)\, dy.
\end{aligned}
$$

The fact that df is exact means its coefficient of dy is zero, i.e. $\psi^1(x,y)\psi^2_y(x,y) = x$. Therefore $\psi^1(-x,y)\psi^2_y(-x,y) = -x$, hence $d\widehat{f}(-x,y) = df(x,y)$.

The orbits of $\widehat{\psi}$ correspond to those of ψ by

$$
\widehat{\psi}^n(x,y) = (x,y) \Leftrightarrow \psi^n(-x,y) = (-x,y),
$$

and we have shown that under this correspondence the values of the action function correspond, up to a difference of $y_- - y_+$ (because of the normalization of $\widehat{f}$). The Calabi invariants similarly differ by $y_- - y_+$, therefore if the conclusion [2, (1.5)] holds for $\widehat{\psi}$ then it holds for ψ. $\square$

The rest of the paper holds as written; note that we no longer have to worry about lifting the requirement $y_+ - y_- \in \mathbb{Z}$, as that is not a hypothesis of Proposition 2.1 (though it is of its analogue [2, Prop. 3.1]).

References

[1] Michael Hutchings. Mean action and the Calabi invariant. *J. Mod. Dyn.*, 10:511–539, 2016.

[2] Morgan Weiler. Mean action of periodic orbits of area-preserving annulus diffeomorphisms. *J. Topol. Anal.*, 13(4), 2021.