Connelly's Proof of Brown's Collaring Theorem

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This brief note is a very close recounting of Connelly's paper "A New Proof of Brown's Collaring Theorem" with some small notational adjustments inserted, written (as most of my notes are) primarily to support my own understanding.

We will write "map" to mean "continuous function." Throughout this note, we fix a Hausdorff space X and a closed subspace $B \subseteq X$.

Definition 1. *If $U \subseteq B$ is open, a **local collar** on U is a closed embedding $h : \bar{U} \times [0,1] \to X$ such that:*

- $h\big(U \times [0,1)\big)$ *is open in X;*
- $h(x,0) = x$ *for every* $x \in \bar{U}$*;*
- *and* $h^{-1}(B) = \bar{U} \times \{0\}$.

If such an h exists, we will say that U admits a local collar. If there is an open cover of B by sets admitting local collars, we will say that B is locally collarable. If B itself admits a local collar, we will say that B is collarable (in this case, notice that the third condition for a local collar is implied by the second).

We define a quotient $\pi : X \sqcup \big(B \times [-1,0]\big) \longrightarrow X^+$ by identifying each point $x \in B \subseteq X$ with $(x,0) \in B \times [-1,0]$. This is illustrated by a pushout diagram:

$$\begin{tikzcd} B \arrow[r, hook, "(-{,}0)"] \arrow[d, hook] & B \times [-1,0] \arrow[d, hook, "\jmath"] \\ X \arrow[r, hook, "\imath"] & X^+ \end{tikzcd}$$

Lemma 2. *The inclusions $\imath$ and $\jmath$ are closed embeddings.*

Proof. If C is closed in X, then $(C \cap B) \times \{0\}$ is closed in $B \times [-1,0]$ and thus

$$\pi^{-1}\big(\imath(C)\big) = C \sqcup \big((C \cap B) \times \{0\}\big)$$

is closed in $X \sqcup \big(B \times [-1,0]\big)$. It follows that $\imath(C)$ is closed in X^+.

If D is closed in $B \times [-1,0]$, then we can write $D \cap \big(B \times \{0\}\big) = C \times \{0\}$, where C is closed in B. But B is closed in X, so C is closed in X and thus

$$\pi^{-1}\big(\jmath(D)\big) = C \sqcup D$$

is closed in $X \sqcup \big(B \times [-1,0]\big)$. It follows that $\jmath(D)$ is closed in X^+. $\square$

If $h : \bar{U} \times [0,1] \to X$ is a local collar on U, we have two closed embeddings:

$$\imath \circ h : \bar{U} \times [0,1] \to X^+$$

$$\jmath : \bar{U} \times [-1,0] \to X^+$$

The two domains form a closed cover of $\bar{U} \times [-1,1]$ and the two maps agree on the intersection of the domains, since $\jmath(x,0) = \imath(x) = \imath \circ h(x,0)$ for any $x \in \bar{U}$. Hence, these maps glue to define a closed map $H : \bar{U} \times [-1,1] \to X^+$ given by

$$H(x,t) = \begin{cases} \imath \circ h(x,t), & t \geq 0 \\ \jmath(x,t), & t \leq 0 \end{cases}$$

Lemma 3. *The map $H : \bar{U} \times [-1,1] \to X^+$ is a closed embedding.*

Proof. It remains only to confirm injectivity. Suppose $x, y \in \bar{U}$ and $s, t \in [-1,1]$ satisfy $H(x,s) = H(y,t)$. Without loss of generality, we may assume that $s \leq t$. If $s \geq 0$, then $t \geq 0$ as well, so $(x,s) = (y,t)$ by the injectivity of $\imath \circ h$. If $t \leq 0$, then $s \leq 0$ as well, so $(x,s) = (y,t)$ by the injectivity of $\jmath$. It remains to consider the case where $s < 0 < t$, or rather, to show that it cannot occur. This gives

$$\jmath(x,s) = H(x,s) = H(y,t) = \imath \circ h(y,t) \in \imath(X)$$

and thus $(x,s) \in \jmath^{-1}\big(\imath(X)\big) = B \times \{0\}$, which clearly contradicts $s < 0$. $\square$

Consider a map $f : B \to [-1,0]$. Given $a \in [0,1]$ and a closed subset $C \subseteq B$, we define a closed set $S_f^a(C) \subseteq B \times [-1,a]$ by

$$S_f^a(C) = \big\{(x,t) \in C \times [-1,1] : f(x) \leq t \leq a\big\}.$$

Given a closed embedding $g : X \to X^+$, we will say that (f,g) is a **good pair** if $g(X) = \imath(X) \cup \jmath\big(S_f^0(B)\big)$ and $g(x) = \jmath\big(x, f(x)\big)$ for all $x \in B$.

Lemma 4. *If $f(B) = \{-1\}$ and (f,g) is a good pair, then B is collarable.*

Proof. Since $f(B) = \{-1\}$, we have $S_f^0(B) = B \times [-1,0]$, so g is surjective and thus a homeomorphism. Define $h : B \times [0,1] \to X$ by $h(x,t) = g^{-1} \circ \jmath(x, t-1)$. Then h is a closed embedding by Lemma 2 and for any $x \in B$, we have

$$h(x,0) = g^{-1} \circ \jmath(x,-1) = g^{-1} \circ \jmath\big(x, f(x)\big) = x.$$

Moreover, notice that $B \times [-1,0)$ is open in $X \sqcup B \times [-1,0]$ and saturated with respect to π. Thus $\pi\big(B \times [-1,0)\big) = \jmath\big(B \times [-1,0)\big)$ is open in X^+ and hence $h\big(B \times [0,1)\big)$ is open in X. This proves that h is a local collar on B. $\square$

We are now almost in a position to prove the main theorem by induction, but for convenience, we package the inductive step seperately in one last lemma:

Lemma 5. *Let B be compact and let $h : \bar{U} \times [0,1] \to X$ be a local collar on U. If (f,g) is a good pair and $C \subseteq U$ is closed in B, then there exists another good pair (f_+, g_+) such that $f_+ \leq f$ and $f_+(C) = \{-1\}$.*

Proof. Since B is compact Hausdorff, Urysohn's lemma guarantees the existence of a map $\lambda : B \to [0,1]$ with $\lambda(C) = \{1\}$ and $\lambda(B - U) = \{0\}$. We then define a map $f_+ : B \to [-1,0]$ by

$$f_+(x) = \big(1 - \lambda(x)\big) f(x) - \lambda(x)$$

Notice that we have expressed $f_+(x)$ as a convex combination of $f(x)$ and -1. Since $-1 \le f \le 0$, this implies that $-1 \le f_+ \le f \le 0$. For any $x \in C$, we have $\lambda(x) = 1$ and thus $f_+(x) = -1$. Therefore $f_+(C) = \{-1\}$. For any $x \in B - U$, we have $\lambda(x) = 0$ and thus $f_+(x) = f(x)$.

We define a map $\phi : S^1_f(B) \to S^1_{f_+}(B)$ by

$$\phi(x,t) = \left(x, 1 + \frac{f_+(x) - 1}{f(x) - 1}(t-1)\right)$$

For fixed $x \in B$, the second component of $\phi(x,t)$ is the unique order-preserving, linear bijection $\big[f(x), 1\big] \to \big[f_+(x), 1\big]$. We can see that ϕ is a homeomorphism, because we can explicitly write down its inverse:

$$\phi^{-1}(x,t) = \left(x, 1 + \frac{f(x) - 1}{f_+(x) - 1}(t-1)\right)$$

Given $y \in B$, we have $\phi(y,1) = (y,1)$. Given $x \in B - U$, we have $f_+(x) = f(x)$ and thus $\phi(x,t) = (x,t)$. Thus ϕ restricts to the identity on the set

$$S^1_f(B - U) \cup \big(B \times \{1\}\big) = S^1_{f_+}(B - U) \cup \big(B \times \{1\}\big),$$

where these two sets are equal because $f = f_+$ on $B - U$.

Given any map $\eta : B \to [-1,0]$, we define the following sets:

$$K^1_\eta = H\big(S^1_\eta(\bar{U})\big), \quad K^2 = \imath\Big(X - h\big(U \times [0,1)\big)\Big), \quad K^3_\eta = \jmath\big(S^0_\eta(B - U)\big).$$

These are closed subsets of X^+ by straightforward application of Definition 1, Lemma 2 and Lemma 3. We next verify certain properties of these three sets (the proofs can be skipped without loss of continuity in the broader argument).

(a) $K^1_\eta \cup K^2 \cup K^3_\eta = \imath(X) \cup \jmath\big(S^0_\eta(B)\big)$.

Proof. Notice that $K^2 \subseteq \imath(X)$ and $K^3_\eta \subseteq \jmath\big(S^0_\eta(B)\big)$, as well as

$$K^1_\eta = H\big(S^1_\eta(\bar{U})\big) = H\big(\bar{U} \times [0,1]\big) \cup H\big(S^0_\eta(\bar{U})\big) \subseteq \imath(X) \cup \jmath\big(S^0_\eta(B)\big).$$

Thus we have $K^1_\eta \cup K^2 \cup K^3_\eta \subseteq \imath(X) \cup \jmath\big(S^0_\eta(B)\big)$. To see that this inclusion is in fact equality, notice that

$$\begin{aligned} \imath(X) - K^2 &= \imath(X) - \imath\Big(X - h\big(U \times [0,1)\big)\Big) \\ &\subseteq \imath \circ h\big(U \times [0,1)\big) = H\big(U \times [0,1)\big) \subseteq H\big(S^1_\eta(\bar{U})\big) = K^1_\eta; \\ \jmath\big(S^0_\eta(B)\big) - K^3_\eta &= \jmath\big(S^0_\eta(B)\big) - \jmath\big(S^0_\eta(B - U)\big) \\ &\subseteq \jmath\big(S^0_\eta(\bar{U})\big) = H\big(S^0_\eta(\bar{U})\big) \subseteq H\big(S^1_\eta(\bar{U})\big) = K^1_\eta. \end{aligned}$$

□

(b) $g(X) = K_f^1 \cup K^2 \cup K_f^3$ and $K_f^3 = K_{f_+}^3$.

Proof. By observation (a) and the fact that (f, g) is a good pair, we have

$$g(X) = \imath(X) \cup \jmath\big(S_f^0(B)\big) = K_f^1 \cup K^2 \cup K_f^3.$$

The fact that $K_f^3 = K_{f_+}^3$ follows from the fact that $f = f_+$ on $B - U$. $\square$

(c) $H^{-1}\big(K_\eta^1 \cap (K^2 \cup K_\eta^3)\big) \subseteq S_\eta^1(B - U) \cup \big(B \times \{1\}\big)$.

Proof. Since H is injective, it is equivalent to show that both $K_\eta^1 \cap K^2$ and $K_\eta^1 \cap K_\eta^3$ are contained in $H\Big(S_\eta^1(B - U) \cup \big(B \times \{1\}\big)\Big)$. To show this, we use the fact that intersections and set differences both commute with the image under an injection:

$$\begin{aligned}
K_\eta^1 \cap K^2 &= H\big(S_\eta^1(\bar{U})\big) \cap \imath\Big(X - h\big(U \times [0,1)\big)\Big) \\
&= H\big(S_\eta^1(\bar{U})\big) \cap \imath(X) - \imath \circ h\big(U \times [0,1)\big) \\
&= H\big(\bar{U} \times [0,1]\big) - H\big(U \times [0,1)\big) \\
&= H\big(\bar{U} \times [0,1] - U \times [0,1)\big) \\
&\subseteq H\Big(S_\eta^1(B - U) \cup \big(B \times \{1\}\big)\Big);
\end{aligned}$$

$$\begin{aligned}
K_\eta^1 \cap K_\eta^3 &= H\big(S_\eta^1(\bar{U})\big) \cap \jmath\big(S_\eta^0(B - U)\big) \\
&= H\big(S_\eta^1(\bar{U})\big) \cap H\big(S_\eta^0(B - U)\big) \\
&= H\big(S_\eta^1(\bar{U}) \cap S_\eta^0(B - U)\big) \\
&= H\big(S_\eta^0(\bar{U} - U)\big) \subseteq H\big(S_\eta^1(B - U)\big).
\end{aligned}$$ $\square$

(d) The map ϕ restricts to the identity on the sets

$$H^{-1}\big(K_f^1 \cap (K^2 \cup K_f^3)\big) \qquad \text{and} \qquad H^{-1}\big(K_{f_+}^1 \cap (K^2 \cup K_{f_+}^3)\big).$$

Proof. This follows from observation (c) and the fact ϕ restricts to the identity on the set $S_f^1(B - U) \cup \big(B \times \{1\}\big) = S_{f_+}^1(B - U) \cup \big(B \times \{1\}\big)$. $\square$

By Lemma 3, we have a closed embedding

$$H \circ \phi \circ H^{-1} : K_f^1 \longrightarrow X^+$$

By observation (d), this embedding agrees with the inclusion

$$K^2 \cup K_f^3 \hookrightarrow X^+$$

on the intersection between their respective domains. These two maps are closed embeddings and $\{K_f^1, K^2 \cup K_f^3\}$ is a closed cover of $g(X)$, by observation (b). Hence, these two maps glue to define a closed map $\Phi : g(X) \to X^+$ given by

$$\Phi(x) = \begin{cases} H \circ \phi \circ H^{-1}(x), & x \in K_f^1 \\ x, & x \in K^2 \cup K_f^3 \end{cases}$$

We now verify a couple properties of Φ (as above, the proofs can be skipped without loss of continuity in the broader argument).

(e) Φ is a closed embedding.

Proof. We have already seen that Φ is a closed map, so it remains to verify that Φ is injective. Since both of the piecewise components are injective, it suffices to consider $x \in K_f^1$ and $y \in K^2 \cup K_f^3$ with

$$y = \Phi(y) = \Phi(x) = H \circ \phi \circ H^{-1}(x)$$

and show that $x = y$. Notice that

$$\begin{aligned} x \in K_f^1 &\Longrightarrow H^{-1}(x) \in S_f^1(\bar{U}) \\ &\Longrightarrow \phi \circ H^{-1}(x) \in S_{f_+}^1(\bar{U}) \\ &\Longrightarrow y = H \circ \phi \circ H^{-1}(x) \in K_{f_+}^1 \end{aligned}$$

We have seen that $K_f^3 = K_{f_+}^3$ in observation (b), so $y \in K_{f_+}^1 \cap (K^2 \cup K_{f_+}^3)$. By observation (d), it follows that $\phi \circ H^{-1}(y) = H^{-1}(y) = \phi \circ H^{-1}(x)$. Therefore $y = x$, because $\phi \circ H^{-1}$ is injective. □

(f) $\Phi \circ g(X) = \imath(X) \cup \jmath\big(S_{f_+}^0(B)\big)$.

Proof. We clearly have $\Phi(K^2 \cup K_f^3) = K^2 \cup K_f^3$ and we can calculate

$$\Phi(K_f^1) = H \circ \phi \circ H^{-1}(K_f^1) = H \circ \phi\big(S_f^1(\bar{U})\big) = H\big(S_{f_+}^1(\bar{U})\big) = K_{f_+}^1$$

Then observations (a) and (b) imply that

$$\begin{aligned} \Phi \circ g(X) &= \Phi(K_f^1 \cup K^2 \cup K_f^3) = K_{f_+}^1 \cup K^2 \cup K_f^3 \\ &= K_{f_+}^1 \cup K^2 \cup K_{f_+}^3 = \imath(X) \cup \jmath\big(S_{f_+}^0(B)\big). \end{aligned}$$ □

Since Φ and g are both closed embeddings, we can define another closed embedding by $g_+ = \Phi \circ g : X \to X^+$. If $x \in \bar{U}$, then $H\big(x, f(x)\big) \in K_f^1$ and thus

$$\begin{aligned} g_+(x) &= \Phi \circ g(x) = \Phi \circ \jmath\big(x, f(x)\big) \\ &= \Phi \circ H\big(x, f(x)\big) = H \circ \phi\big(x, f(x)\big) \\ &= H\big(x, f_+(x)\big) = \jmath\big(x, f_+(x)\big). \end{aligned}$$

Meanwhile, if $x \in B - U$, then $\jmath\big(x, f(x)\big) \in K_f^3$ and $f_+(x) = f(x)$, so we have

$$g_+(x) = \Phi \circ g(x) = \Phi \circ \jmath\big(x, f(x)\big) = \jmath\big(x, f(x)\big) = \jmath\big(x, f_+(x)\big).$$

Thus $g_+(x) = \jmath\big(x, f_+(x)\big)$ for all $x \in B$. Along with observation (f), this shows that (f_+, g_+) is a good pair, as desired. $\square$

Having done the hard work of proving the inductive step, the theorem that we have been working towards now follows in a fairly straightforward fashion:

Theorem 6. *If B is compact and locally collarable, then B is collarable.*

Proof. Since B is compact and locally collarable, there exists a finite open cover $\{U_1, \dots, U_n\}$ by sets that admit local collars. Because B is compact Hausdorff, there exists a refined open cover $\{V_1, \dots, V_n\}$ with $\bar{V}_i \subseteq U_i$ for each $i = 1 \dots, n$. For $i = 0, \dots, n$, we inductively define good pairs (f_i, g_i), as follows. Let $g_0 = \imath$ and $f_0 = 0$. Note that (f_0, g_0) is a good pair, because

$$g_0(X) = \imath(X) = \imath(X) \cup \jmath\big(B \times \{0\}\big) = \imath(X) \cup \jmath\big(S^0_{f_0}(B)\big)$$

and $g_0(x) = \imath(x) = \jmath(x, 0) = \jmath\big(x, f_0(x)\big)$ for any $x \in B$. For each $i = 1, \dots, n$, since U_i admits a local collar and $\bar{V}_i \subseteq U_i$ is closed in B, we apply Lemma 5 to the good pair (f_{i-1}, g_{i-1}) to get a good pair (f_i, g_i), which satisfies $f_i \leq f_{i-1}$ and $f_i(\bar{V}_i) = \{-1\}$. In total, we then have $f_1 \geq f_2 \geq \cdots \geq f_n \geq -1$ and since B is covered by $\{V_1, \dots, V_n\}$, it follows that $f_n(B) = \{-1\}$. But (f_n, g_n) is a good pair by construction, so Lemma 4 implies that B is collarable. $\square$