Euler Class and Morse Theory

I will briefly spell out some details of a Morse-theoretic proof that the Euler class of a (co)tangent bundle is Poincaré dual to the Euler characteristic. Consider a closed n-manifold M and a smooth map $f : M \to \mathbb{R}$. Consider $df : M \to T^*M$ as a section of the cotangent bundle and let $Z : M \to T^*M$ denote the zero section. Notice that the critical points of f are given precisely by $df(M) \cap Z(M)$. If p is indeed a critical point of f, then we have three distinguished subspaces of $T_{Z(p)}T^*M$: this vector space splits as $T_{Z(p)}T^*M = T_pM \oplus T_p^*M$ (the tangent space to the zero section and the tangent space to the fiber) and we also define $\Gamma = D_p df(T_pM)$. Since df is a section of the bundle $T^*M \to M$, we can see that the projection map

$$T_{Z(p)}T^*M = T_pM \oplus T_p^*M \longrightarrow T_pM$$

restricts to an isomorphism $\Gamma \cong T_pM$, so we can view $\Gamma \subseteq T_pM \oplus T_p^*M$ as the graph of a unique linear map, which we denote $\eta : T_pM \to T_p^*M$. Explicitly, this is defined by $D_p df(v) = v \oplus \eta(v)$. Then the Hessian form $H : T_pM \times T_pM \to \mathbb{R}$ is simply $H(v, w) = \eta(v)(w)$ (compare this to your favorite definition of the Hessian). From this definition of the Hessian, we can see that

$$H \text{ is non-degenerate} \iff \eta \text{ is an isomorphism} \iff \Gamma \text{ is transverse to } T_pM$$

Thus p is a non-degenerate critical point of f if and only if $df(M)$ and $Z(M)$ intersect transversely at $Z(p)$. Considering all critical points, we see that f is Morse if and only if $df(M)$ and $Z(M)$ intersect transversely. Henceforth, we assume that this is the case.

If M is oriented, then we get orientations on $Z(M)$ and $df(M)$, as well as on each cotangent space T_p^*M. We also have an orientation on T^*M (this is true even when M is non-orientable), which is easy to describe at any point $Z(p)$ on the zero section, via the splitting $T_{Z(p)}T^*M = T_pM \oplus T_p^*M$. For any critical point p, we have the subspace $\Gamma = D_p df(T_pM)$ and the projection maps

$$T_pM \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} T_p^*M$$

Since $D_p df = \pi_1^{-1}$ is precisely the map giving Γ its orientation, we can see that π_1 must preserve orientation. It follows that

$$\begin{aligned} T_pM \text{ and } \Gamma \text{ intersect positively} &\iff T_pM \oplus \Gamma \xrightarrow{\mathrm{Id} \oplus \pi_2} T_pM \oplus T_p^*M \text{ preserves orientation} \\ &\iff \pi_2 : \Gamma \longrightarrow T_p^*M \text{ preserves orientation} \\ &\iff \eta = \pi_2 \circ \pi_1^{-1} : T_pM \longrightarrow T_p^*M \text{ preserves orientation} \\ &\iff H \text{ has an even number of negative eigenvalues} \\ &\iff \text{the critical point } p \text{ has even index} \end{aligned}$$

Therefore, if σ_p is the sign of the intersection at $p \in df(M) \cap Z(M)$, then the quantity

$$\chi(M) = \sum_{\mathrm{crit}(f)} (-1)^{\mathrm{index}(p)} = \sum_{\mathrm{crit}(f)} \sigma_p$$

is precisely the intersection number of $df(M)$ and $Z(M)$. When M is connected, we know that this intersection number represents the cap product $e(T^*M) \frown [M] \in H_0(M; \mathbb{Z})$. Thus $e(T^*M)$ and $\chi(M)$ are Poincaré dual. If M is not oriented, we can still give the same arguent with $\mathbb{Z}/2$ coefficients, ignoring all issues of sign.