An Exotic Sphere

Nikhil Sahoo

Here, I will describe Milnor's construction of an exotic smooth structure on S^7. My goal is to write out the details that are elided in [M1]. Since only a couple of years separate the discovery and classification of exotic 7-spheres, I will not worry about the invariant constructed in [M1], but rather follow the heuristic outlined in [M5]. I hope this will make my exposition a little more readable.

Contents

1 Tautological Bundles

1.1 Projective spaces

In this first section, we will work with one of the associative division algebras

$$\mathbb{F} = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}, \text{ with dimension } n = \dim_{\mathbb{R}} \mathbb{F} = 1, 2 \text{ or } 4.$$

Since $\mathbb{H}$ is not commutative, we must be a little careful with our "linear algebra." For our purposes, an $\mathbb{F}$-vector space is a right $\mathbb{F}$-module; the prototype is $\mathbb{F}^k$, where scalars act by component-wise multiplication on the right. Since a linear subspace needs to be invariant under right-multiplication, any non-zero vector $v \in \mathbb{F}^k$ spans a right-line $v\mathbb{F}$. The projective space $\mathbb{FP}^k$ is defined to be the set of right-lines in $\mathbb{F}^{k+1}$. If $v = (v_0, v_1, \ldots, v_k) \in \mathbb{F}^{k+1}$ is non-zero, we write

$$v\mathbb{F} = [v_0 : v_1 : \cdots : v_k].$$

Notice that $[v_0 : v_1 : \cdots : v_k] = [w_0 : w_1 : \cdots : w_k]$ if and only if there exists some scalar $\lambda \in \mathbb{F}$ such that $v_i = w_i\lambda$ for all $i = 0, 1, \ldots, k$. We have an inclusion

$$\mathbb{F}^k \hookrightarrow \mathbb{FP}^k \qquad \text{given by} \qquad (v_1, \ldots, v_k) \mapsto [1 : v_1 : \cdots : v_k].$$

The lines that are *not* in the image of this map are precisely those of the form

$$[0 : v_1 : \cdots : v_k],$$

i.e. lines contained in $\{0\} \times \mathbb{F}^k$. Therefore, we can decompose $\mathbb{FP}^k = \mathbb{F}^k \sqcup \mathbb{FP}^{k-1}$ as a disjoint union[1] of $\mathbb{FP}^{k-1}$ with the open kn-cell $\mathbb{F}^k$. Taking $t \in \mathbb{R}$, we have

$$\lim_{t\to\infty} [1 : v_1 t : \cdots : v_k t] = \lim_{t\to\infty} \left[\tfrac{1}{t} : v_1 : \cdots : v_k\right] = [0 : v_1 : \cdots : v_k].$$

This shows that $\mathbb{FP}^k$ is formed from $\mathbb{FP}^{k-1}$ by the attachment of a closed kn-cell, where the attaching map takes a point $v \in S^{kn-1} \subseteq \mathbb{F}^k$ to the line $v\mathbb{F} \in \mathbb{FP}^{k-1}$. Inductively, we can see that $\mathbb{FP}^k$ is homeomorphic to a CW complex with exactly one cell in each dimension $0, n, 2n, \ldots, kn$. For example, we have

$$\mathbb{FP}^0 = \text{point}, \qquad \mathbb{FP}^1 = S^n, \qquad \mathbb{FP}^2 = D^{2n} \cup_f S^n,$$

where $f : S^{2n-1} \to S^n$ is an attaching map.[2] We can think of $\mathbb{FP}^1 = \mathbb{F} \cup \{\infty\}$ as a one-point compactification of our scalars, by associating a slope $\lambda \in \mathbb{F}$ to

$$[1 : \lambda] = \{(x, y) \in \mathbb{F}^2 : y = \lambda x\} \in \mathbb{FP}^1.$$

[1] This is purely an operation on sets; we are not using the disjoint union topology.

[2] This attaching map has Hopf invariant 1, yielding examples of such a map for $n = 1, 2, 4$. There is a similar construction for $n = 8$ that uses the octonions $\mathbb{O}$ and $\mathbb{OP}^2 = D^{16} \cup_f S^8$, but the definition of $\mathbb{OP}^2$ is complicated by the fact that $\mathbb{O}$ is not associative and so lines are not well-defined in $\mathbb{O}^3$. Indeed, while $\mathbb{OP}^2$ exists, there is no reasonable space that we might call $\mathbb{OP}^k$ when $k > 2$ [FF].

This inclusion misses the line $[0:1] = \{(x,y) \in \mathbb{F}^2 : x = 0\}$ with "infinite slope."

To see that the attaching map $f : S^{kn-1} \to \mathbb{FP}^{k-1}$ is a unit sphere bundle,[3] we consider the tautological bundle $\gamma_k^{\mathbb{F}}$ with total space

$$E(\gamma_k^{\mathbb{F}}) = \{(v,\ell) \in \mathbb{F}^k \times \mathbb{FP}^{k-1} : v \in \ell\}.$$

This space has projections to the factors $\mathbb{F}^k$ and $\mathbb{FP}^{k-1}$, so we get a diagram

$$\begin{array}{ccccc} S(\gamma_k^{\mathbb{F}}) & \hookrightarrow & E(\gamma_k^{\mathbb{F}}) & \longrightarrow & \mathbb{FP}^{k-1} \\ \downarrow & & \downarrow & & \\ S^{kn-1} & \hookrightarrow & \mathbb{F}^k & & \end{array}$$

For a point $(v,\ell) \in S(\gamma_k^{\mathbb{F}})$ in the unit sphere bundle, we have $|v| = 1$ and $\ell = v\mathbb{F}$, so the projection $E(\gamma_k^{\mathbb{F}}) \to \mathbb{F}^k$ restricts to a homeomorphism $S(\gamma_k^{\mathbb{F}}) \to S^{kn-1}$. Under this identification, the other projection $E(\gamma_k^{\mathbb{F}}) \to \mathbb{FP}^{k-1}$ restricts to our attaching map $f(v) = v\mathbb{F}$. As an $\mathbb{R}$-vector bundle, the rank of $\gamma_k^{\mathbb{F}}$ is $n = \dim_{\mathbb{R}} \mathbb{F}$, so our unit sphere bundle[4] has the form

$$S^{n-1} \hookrightarrow S^{kn-1} \to \mathbb{FP}^{k-1}.$$

When $k = 2$, the base space is also a sphere:

$$S^0 \hookrightarrow S^1 \to \mathbb{RP}^1 \cong S^1 \qquad S^1 \hookrightarrow S^3 \to \mathbb{CP}^1 \cong S^2 \qquad S^3 \hookrightarrow S^7 \to \mathbb{HP}^1 \cong S^4$$

These bundles are called the "Hopf fibrations" and we will mainly be concerned with the quaternionic case, where we have now exhibited the standard 7-sphere as a unit sphere bundle over S^4. To build some candidates for exotic 7-spheres, we will use the unit sphere bundles for other vector bundles over S^4. But first, we need to perform one more calculation in the specific case of the bundle $\gamma_2^{\mathbb{H}}$.

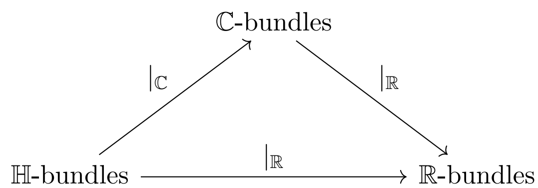
1.2 Calculating $p_1(\gamma_2^{\mathbb{H}})$

Viewing $\gamma_2^{\mathbb{H}}$ as a real vector bundle, we will show that $p_1(\gamma_2^{\mathbb{H}}) = \pm 2\beta$, where β is a generator of $H^4(\mathbb{HP}^1) \cong \mathbb{Z}$ and p_1 is the first Pontryagin class. Our proof is mostly based on this explanation that Jason DeVito gave on StackExchange. The rest can be found in Chapter 14 of [M4] or Chapter 19 of [FF].

The inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ yield the forgetful functors in this diagram:

[3] We say "sphere bundle" to mean a bundle with spherical fibers and "unit sphere bundle" to mean a sphere bundle arising as the space of unit vectors inside of a normed vector bundle.

[4] We can also interpret this as a principal bundle, where the right action $S^{n-1} \circlearrowright S^{kn-1}$ is given by multiplying unit vectors in $S^{kn-1} \subseteq \mathbb{F}^k$ by unit scalars in $S^{n-1} \subseteq \mathbb{F}$ on the right.



Given a real vector bundle η, we can also form a complex vector bundle $\eta \otimes \mathbb{C}$. Composing this operation with the forgetful functor $|_{\mathbb{R}}$ has the following effect:

$$(\eta \otimes \mathbb{C})|_{\mathbb{R}} \cong \eta \oplus \eta \qquad \text{and} \qquad \xi|_{\mathbb{R}} \otimes \mathbb{C} \cong \xi \oplus \bar{\xi},$$

where $\bar{\xi}$ is the conjugate bundle of ξ. The first Pontryagin class of a real vector bundle η is $p_1(\eta) = -c_2(\eta \otimes \mathbb{C})$, so if ξ is a complex vector bundle, then we have

$$\begin{aligned} -p_1(\xi|_{\mathbb{R}}) &= c_2(\xi|_{\mathbb{R}} \otimes \mathbb{C}) = c_2(\xi \oplus \bar{\xi}) \\ &= c_2(\xi)c_0(\bar{\xi}) + c_1(\xi)c_1(\bar{\xi}) + c_0(\xi)c_2(\bar{\xi}) = 2c_2(\xi) - c_1(\xi)^2, \end{aligned}$$

using the Whitney sum formula and $c_i(\bar{\xi}) = (-1)^i c_i(\xi)$. Because $H^2(\mathbb{HP}^1) = 0$, we have $c_1(\gamma_2^{\mathbb{H}}|_{\mathbb{C}}) = 0$ and therefore

$$p_1(\gamma_2^{\mathbb{H}}|_{\mathbb{R}}) = -2c_2(\gamma_2^{\mathbb{H}}|_{\mathbb{C}}).$$

To prove that $p_1(\gamma_2^{\mathbb{H}}|_{\mathbb{R}}) = \pm 2\beta$, it therefore suffices to show that $c_2(\gamma_2^{\mathbb{H}}|_{\mathbb{C}}) = \pm\beta$, i.e. that this Chern class generates $H^4(\mathbb{HP}^1)$.

We may consider $\mathbb{H}^2$ as a 4-dimensional complex vector space, where i acts by right-multiplication. We then view $\mathbb{CP}^3$ as the space of complex lines in $\mathbb{H}^2$ and we define a map $g : \mathbb{CP}^3 \to \mathbb{HP}^1$ by $g(\ell) = \ell\mathbb{H} = \ell \oplus \ell j$. Given any $L \in \mathbb{HP}^1$, we have $g^{-1}(L) = \{\ell \in \mathbb{CP}^3 : \ell \subseteq L\} \cong \mathbb{CP}^1$, since L is 2-dimensional over $\mathbb{C}$. Indeed, we can see that g is a $\mathbb{CP}^1$-bundle by viewing it as the "projectivization" of the complex vector bundle $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$ (any local trivialization of $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$ yields a local trivialization of g). Identifying $S^2 \cong \mathbb{CP}^1$, we have an oriented sphere bundle, where the orientation comes from the complex structure on each fiber. Hence, we have a Gysin sequence, which includes the following useful isomorphism:

$$0 = H^1(\mathbb{HP}^1) \longrightarrow H^4(\mathbb{HP}^1) \xrightarrow{g^*} H^4(\mathbb{CP}^3) \longrightarrow H^2(\mathbb{HP}^1) = 0$$

Thus $c_2(\gamma_2^{\mathbb{H}}|_{\mathbb{C}})$ generates $H^4(\mathbb{HP}^1)$ if and only if $g^* c_2(\gamma_2^{\mathbb{H}}|_{\mathbb{C}})$ generates $H^4(\mathbb{CP}^3)$. Below, we will show that $g^*\gamma_2^{\mathbb{H}}|_{\mathbb{C}} \cong \gamma_3^{\mathbb{C}} \oplus \overline{\gamma_3^{\mathbb{C}}}$, so the Whitney sum formula gives

$$\begin{aligned} g^* c_2(\gamma_2^{\mathbb{H}}|_{\mathbb{C}}) &= c_2(g^*\gamma_2^{\mathbb{H}}|_{\mathbb{C}}) = c_2\big(\gamma_3^{\mathbb{C}} \oplus \overline{\gamma_3^{\mathbb{C}}}\big) \\ &= c_0(\gamma_3^{\mathbb{C}})c_2\big(\overline{\gamma_3^{\mathbb{C}}}\big) + c_1(\gamma_3^{\mathbb{C}})c_1\big(\overline{\gamma_3^{\mathbb{C}}}\big) + c_2(\gamma_3^{\mathbb{C}})c_0\big(\overline{\gamma_3^{\mathbb{C}}}\big) = -c_1(\gamma_3^{\mathbb{C}})^2 \end{aligned}$$

This class indeed generates $H^4(\mathbb{CP}^3)$, since we know that $H^*(\mathbb{CP}^3) \cong \mathbb{Z}[a]/(a^4)$ with the generator a corresponding to $c_1(\gamma_3^{\mathbb{C}})$. It remains to prove the promised isomorphism of complex vector bundles over $\mathbb{CP}^3$. To do so, we define a mapping

$$\begin{array}{ccc} E(\gamma_3^{\mathbb{C}} \oplus \gamma_3^{\mathbb{C}}) & \overset{G}{\dashrightarrow} & E(\gamma_2^{\mathbb{H}}) \\ \downarrow & & \downarrow \\ \mathbb{CP}^3 & \xrightarrow{\quad g \quad} & \mathbb{HP}^1 \end{array}$$

by sending a pair of vectors $v, w \in \ell$ (with $\ell \in \mathbb{CP}^3$) to the vector $v + wj \in \ell \oplus \ell j$. The map G is an $\mathbb{R}$-linear isomorphism on fibers; because i and j anti-commute, we can see that G is $\mathbb{C}$-linear in v and $\mathbb{C}$-antilinear in w. Therefore, it induces an isomorphism $\gamma_3^{\mathbb{C}} \oplus \overline{\gamma_3^{\mathbb{C}}} \cong g^* \gamma_2^{\mathbb{H}}|_{\mathbb{C}}$, as desired.

2 Candidate Spaces

2.1 The clutching construction

In this section, we outline a method of describing oriented vector bundles over a sphere. Choosing a pair of antipodes $p_\pm \in S^k$, we can write $S^k = \mathbb{R}^k_+ \cup \mathbb{R}^k_-$, where the sets $\mathbb{R}^k_\pm = S^k \setminus p_\pm$ are identified with k-dimensional Euclidean space. Explicitly, these two copies of $\mathbb{R}^k$ are glued by the map

$$\begin{array}{ccc} \mathbb{R}^k_+ \backslash 0 & \longrightarrow & \mathbb{R}^k_- \backslash 0 \\ u & \longmapsto & u/|u|^2 \end{array}$$

This becomes even simpler if we restrict to unit disks and write $S^k = D^k_+ \cup D^k_-$, where the two disks are glued by the identity $S^{k-1}_+ = S^{k-1}_-$ on their boundaries. In what follows, we will extend these gluings to define vector bundles over S^k.

Given a smooth map $\varphi : S^{k-1} \to \mathrm{SO}(n)$, which is called a clutching function, we can define a vector bundle over S^k by gluing together trivial bundles over $\mathbb{R}^k_\pm$ in a way that lifts the above gluing and uses φ to define a transition map:

$$\begin{array}{l} E = (\mathbb{R}^k_+ \times \mathbb{R}^n) \cup (\mathbb{R}^k_- \times \mathbb{R}^n) \\ \downarrow \\ S^k = \mathbb{R}^k_+ \cup \mathbb{R}^k_- \end{array}$$

Concretely, the gluing is defined by the following map:

$$\begin{array}{ccc} \mathbb{R}^k_+ \backslash 0 \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^k_- \backslash 0 \times \mathbb{R}^n \\ (u, v) & \longmapsto & \left(\dfrac{u}{|u|^2}, \varphi\left(\dfrac{u}{|u|}\right) v \right) \end{array}$$

Because φ is smooth and takes values in the orientation-preserving isomorphisms of the vector space $\mathbb{R}^n$, we can see that this defines a smooth, oriented $\mathbb{R}^n$-bundle over S^k (note that the gluing is over an open subset of S^k). We can also restrict to the gluing $S^k = D^k_+ \cup D^k_-$ to give an equivalent definition of the total space:

$$\begin{array}{l} E = (D^k_+ \times \mathbb{R}^n) \cup (D^k_- \times \mathbb{R}^n) \\ \big\downarrow \\ S^k = D^k_+ \cup D^k_- \end{array}$$

This is defined by a simple gluing map, to which we give more explicit notation:

$$\begin{aligned} \overline{\varphi} : S^{k-1}_+ \times \mathbb{R}^n &\longrightarrow S^{k-1}_- \times \mathbb{R}^n \\ (u, v) &\longmapsto \big(u, \varphi(u)v\big) \end{aligned}$$

We introduce both frameworks, because the latter is often easier to work with, but the former better elucidates the smooth structure and local trivializability.

Suppose that we have a homotopy $\Phi : S^{k-1} \times I \to \mathrm{SO}(n)$, where $I = [0, 1]$ is the unit interval. We can repeat this process with an extra parameter $t \in I$, to get an oriented $\mathbb{R}^n$-bundle over $S^k \times I$. Explicitly, we have another a gluing:

$$\begin{array}{l} E = (\mathbb{R}^k_+ \times I \times \mathbb{R}^n) \cup (\mathbb{R}^k_- \times I \times \mathbb{R}^n) \\ \big\downarrow \\ S^k \times I = (\mathbb{R}^k_+ \times I) \cup (\mathbb{R}^k_- \times I) \end{array}$$

The gluing defining the total space of this bundle is given by:

$$\begin{aligned} \mathbb{R}^k_+ \backslash 0 \times I \times \mathbb{R}^n &\longrightarrow \mathbb{R}^k_- \backslash 0 \times I \times \mathbb{R}^n \\ (u, t, v) &\longmapsto \left(\frac{u}{|u|^2}, t, \varphi\left(\frac{u}{|u|}, t\right)v\right) \end{aligned}$$

To get the gluing defining the base space, we simply ignore the last component. Let $\xi : E \to S^k \times I$ denote the bundle so defined and let $\iota_t : S^k \hookrightarrow S^k \times I$ denote the inclusion of $S^k \times \{t\} \subseteq S^k \times I$. Then the restricted bundle $\iota_t^* \xi$ is clearly given by the clutching function $\Phi \circ \iota_t$. The homotopy-invariance of pullback bundles yields $\iota_0^* \xi \cong \iota_1^* \xi$, so the isomorphism class of our oriented $\mathbb{R}^n$-bundle depends only on the homotopy class of the clutching function. Therefore, the clutching construction defines a function[5] of the form

$$\pi_{k-1}\mathrm{SO}(n) \longrightarrow \left\{ \begin{array}{c} \text{isomorphism} \\ \text{classes of oriented} \\ \mathbb{R}^n\text{-bundles over } S^k \end{array} \right\}$$

[5] In fact, this function is a bijection [A].

Since the clutching construction yields oriented bundles, we might ask how to modify a clutching function to get the same bundle with opposite orientation. Consider a clutching function $\varphi : S^{k-1} \to \mathrm{SO}(n)$ and an orientation-reversing transformation $T \in \mathrm{O}(n) \setminus \mathrm{SO}(n)$. We can conjugate to define another clutching function $\varphi' = T\varphi T^{-1}$. Let E and E' be the total spaces of the vector bundles corresponding respectively to φ and φ'. Consider the following bundle maps:

$$\begin{array}{ccc} D^k_+ \times \mathbb{R}^n & \longrightarrow & D^k_+ \times \mathbb{R}^n \\ (u,v) & \longmapsto & \big(u, Tv\big) \end{array} \qquad \begin{array}{ccc} D^k_- \times \mathbb{R}^n & \longrightarrow & D^k_- \times \mathbb{R}^n \\ (u,v) & \longmapsto & \big(u, Tv\big) \end{array}$$

These maps clearly reverse the orientation of each fiber. To see that they glue together to form an orientation-reversing vector bundle isomorphism $E \to E'$, simply note that the following diagram commutes by the definition of φ':

$$\begin{array}{ccc} S^{k-1}_+ \times \mathbb{R}^n & \xrightarrow{\overline{\varphi}} & S^{k-1}_- \times \mathbb{R}^n \\ \downarrow{\scriptstyle \mathrm{Id}\times T} & & \downarrow{\scriptstyle \mathrm{Id}\times T} \\ S^{k-1}_+ \times \mathbb{R}^n & \xrightarrow{\overline{\varphi}'} & S^{k-1}_- \times \mathbb{R}^n \end{array}$$

Therefore, the clutching functions φ and $\varphi' = T\varphi T^{-1}$ define the same bundle, but with opposite orientations. We will make use of this fact in §2.4 below.

While describing this construction, we have really belabored the smoothness, because the whole point of these notes is to construct certain smooth manifolds. But there is a more general clutching construction, which starts with a compact Hausdorff space X and uses a clutching function $\varphi : X \to \mathrm{SO}(n)$ to construct an oriented $\mathbb{R}^n$-bundle over the reduced suspension ΣX. To form this bundle, we start with trivial bundles over two reduced cones $CX_\pm$ and lift the gluing $\Sigma X = CX_+ \cup_X CX_-$ to a gluing of fibers:

$$\begin{array}{l} E = (CX_+ \times \mathbb{R}^n) \cup (CX_- \times \mathbb{R}^n) \\ \downarrow \\ \Sigma X = CX_+ \cup CX_- \end{array}$$

Just as before, the gluing of the total space is given by a straightforward map:

$$\begin{array}{rcl} \overline{\varphi} : X_+ \times \mathbb{R}^n & \longrightarrow & X_- \times \mathbb{R}^n \\ (u,v) & \longmapsto & \big(u, \varphi(u)v\big) \end{array}$$

We will not repeat all of the details in this setting, but suffice to note that this construction is natural, in the following sense. Suppose Y is another compact

Hausdorff space and $g : Y \to X$ is a continuous map. Then we can define vector bundles ξ and η over ΣX and ΣY, using the clutching functions φ and $\varphi \circ g$, respectively. Using the suspended map $\Sigma g : \Sigma Y \to \Sigma X$, we can then write down an isomorphism $\eta \cong \Sigma g^* \xi$. This isomorphism is defined by a bundle map, which is an isomorphism on every fiber:

$$\begin{array}{ccc} E(\eta) & \dashrightarrow & E(\xi) \\ \downarrow & & \downarrow \\ \Sigma Y & \xrightarrow{\Sigma g} & \Sigma X \end{array}$$

To define the map between the total spaces, we start with the maps

$$Cg \times \mathrm{Id}_{\mathbb{R}^n} : CY_\pm \times \mathbb{R}^n \longrightarrow CX_\pm \times \mathbb{R}^n$$

and check that they are compatible with the gluings that define $E(\eta)$ and $E(\xi)$:

$$\begin{array}{ccc} Y_+ \times \mathbb{R}^n & \xrightarrow{g \times \mathrm{Id}_{\mathbb{R}^n}} & X_+ \times \mathbb{R}^n \\ {\scriptstyle \overline{\varphi \circ g}}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}} \\ Y_- \times \mathbb{R}^n & \xrightarrow{g \times \mathrm{Id}_{\mathbb{R}^n}} & X_- \times \mathbb{R}^n \end{array}$$

A more detailed explanation of the general construction can be found in [A].[6]

2.2 A sample clutching function

We now describe a clutching function for the bundle $\gamma_2^{\mathbb{H}}$ over S^4. In §2.4 below, we will see how our knowledge of this clutching function and the corresponding Pontryagin class yields knowledge of various other Pontryagin classes.

If $D^4_\pm \subseteq \mathbb{H}$ are two copies of the unit disk, we can write the decomposition $\mathbb{HP}^1 \cong S^4 = D^4_+ \cup D^4_-$ very concretely, via the inclusion maps:

$$\begin{array}{ccc} D^4_+ & \longrightarrow & \mathbb{HP}^1 \\ u & \longmapsto & [1 : u] \end{array} \qquad \begin{array}{ccc} D^4_- & \longrightarrow & \mathbb{HP}^1 \\ u & \longmapsto & [\bar{u} : 1] \end{array}$$

[6] Given a topological group G and a compact Hausdorff space X, we let $\mathrm{Bun}_G(X)$ denote the set of isomorphism classes of principle G-bundles over X. Then the clutching construction yields a bijection $[X, G] = \mathrm{Bun}_G(\Sigma X)$, where each side is functorial and the correspondence between them is a natural transformation. The set $\mathrm{Bun}_G(\Sigma X)$ is also classified by $[\Sigma X, BG]$. The two descriptions are related by the weak equivalence $G \simeq \Omega BG$ and the adjunction $\Sigma \dashv \Omega$:

$$[X, G] = [X, \Omega BG] = [\Sigma X, BG].$$

For a unit quaternion $u \in S^3$, the quaternionic conjugate $\bar{u}$ is also the inverse. This implies that $[1 : u] = [u^{-1} : 1] = [\bar{u} : 1]$, so the two inclusions indeed agree on the unit sphere. We now define a function $\varphi : S^3 \to \mathrm{SO}(4)$ given by

$$\varphi(u) : v \mapsto uv,$$

where we view both u and v as quaternions. Then this clutching function defines an $\mathbb{R}^4$-bundle $\xi : E \to \mathbb{HP}^1$, which is isomorphic to the tautological bundle $\gamma_2^{\mathbb{H}}$. To see this, consider the following diagram:

$$\begin{array}{cccccc}
 & & (u,v) & \longmapsto & \big((v, uv), [1 : u]\big) & \\
(u,v) & S^3_+ \times \mathbb{H} & \hookrightarrow & D^4_+ \times \mathbb{H} & \longrightarrow & E(\gamma_2^{\mathbb{H}}) \\
\downarrow & \downarrow \bar{\varphi} & & & & \| \\
(u,uv) & S^3_- \times \mathbb{H} & \hookrightarrow & D^4_- \times \mathbb{H} & \longrightarrow & E(\gamma_2^{\mathbb{H}}) \\
 & & (u', v') & \longmapsto & \big((\bar{u}'v', v'), [\bar{u}' : 1]\big) &
\end{array}$$

This diagram commutes because $(\bar{u}uv, uv) = (v, uv)$ for any $u \in S^3$ and $v \in \mathbb{H}$. Since the map $\bar{\varphi} : S^3_+ \times \mathbb{H} \to S^3_- \times \mathbb{H}$ is the gluing used to define the bundle E, the above trivializations of $\gamma_2^{\mathbb{H}}$ over $D^4_\pm \subseteq \mathbb{HP}^1$ glue to form a map $E \to E(\gamma_2^{\mathbb{H}})$, which is an isomorphism $\xi \cong \gamma_2^{\mathbb{H}}$.

2.3 The homomorphism property

Because $\pi_{k-1}\mathrm{SO}(n)$ is a group whenever $k > 1$, we might ask how the clutching construction reflects this group structure. Unfortunately, there is no obvious group structure on isomorphism classes of bundles with which we can compare.[7] But if we fix a characteristic class $c \in \widetilde{H}^*\big(G_n(\mathbb{R}^\infty); \mathbb{Z}\big)$, then the composition

$$\pi_{k-1}\mathrm{SO}(n) \xrightarrow{\text{clutching}} \left\{ \begin{array}{c} \text{isomorphism} \\ \text{classes of oriented} \\ \mathbb{R}^n\text{-bundles over } S^k \end{array} \right\} \xrightarrow{\quad c \quad} \widetilde{H}^*(S^k)$$

is a homomorphism whenever $k > 1$, as we will now prove.[8]

[7] It is possible to define a monoid structure by extending the connected sum $S^k \# S^k = S^k$ to the total spaces of the vector bundles. In terms of this structure, the clutching construction yields a group isomorphism. However, we do not want to go through the tedium of defining this structure and verifying well-definedness, before we compare it to the clutching construction.

[8] We state this in terms of a characteristic class of unoriented bundles, because we will only need the first Pontryagin class p_1. However, our clutching functions also define an orientation

In proving this, all of our spaces will implicitly be pointed. To begin with, let $\Psi : S^{k-1} \to S^{k-1} \vee S^{k-1}$ be the quotient defined by collapsing an equator $S^{k-2} \subseteq S^{k-1}$ down to a point. Applying the reduced suspension functor yields an analogous map $\Sigma\Psi : S^k \to S^k \vee S^k$, which collapses the suspended equator

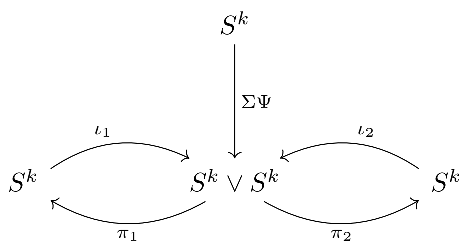
$$\Sigma S^{k-2} \subseteq \Sigma S^{k-1} = S^k$$

down to a point. Here, we are using the fact that $\Sigma(S^{k-1} \vee S^{k-1}) = S^k \vee S^k$.

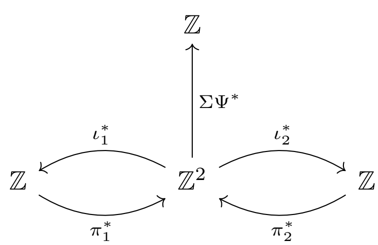
Let $\iota_1, \iota_2 : S^k \hookrightarrow S^k \vee S^k$ be the inclusions of the two spheres into the wedge sum and let $\pi_1, \pi_2 : S^k \vee S^k \to S^k$ be the quotients collapsing one of the spheres down to a point. We label these maps so that

$$\pi_1 \circ \iota_1 = \mathrm{Id}_{S^k} \qquad \pi_1 \circ \iota_2 = \text{constant}$$
$$\pi_2 \circ \iota_2 = \mathrm{Id}_{S^k} \qquad \pi_2 \circ \iota_1 = \text{constant}$$

The maps $\pi_j \circ \Sigma\Psi : S^k \to S^k$ are given by collapsing one of the hemispheres bounded by ΣS^{k-2} to a point; these compositions are both homotopic to Id_{S^k}. All of these maps are illustrated in the following diagram:



Applying the reduced singular cohomology functor, this diagram becomes:



The homomorphisms ι_1^* and ι_2^* are the natural projections out of the product, by the Eilenberg-Steenrod axiom of additivity. Then we can see that π_1^* and π_2^*

and we could just as well have chosen a characteristic class that does depend on orientation, such as the Euler class e. Moreover, we could have used a characteristic class valued in any (additive) reduced cohomology theory. In particular, we can define a homomorphism

$$\pi_{k-1}\mathrm{SO}(n) \to \widetilde{\mathrm{KO}}(S^k)$$

by taking a vector bundle to its equivalence class. This is an isomorphism for all $n > k > 1$.

are the natural inclusions into the direct sum, because

$$\iota_1^* \circ \pi_1^* = \mathrm{Id}_{\mathbb{Z}} \qquad \iota_1^* \circ \pi_2^* = 0$$
$$\iota_2^* \circ \pi_2^* = \mathrm{Id}_{\mathbb{Z}} \qquad \iota_2^* \circ \pi_1^* = 0$$

We can now confirm that the map $\Sigma\Psi^* : \mathbb{Z}^2 \to \mathbb{Z}$ simply adds the two integers:

$$\begin{aligned}\Sigma\Psi^*(v) &= \Sigma\Psi^*\big(\pi_1^*\iota_1^*(v) + \pi_2^*\iota_2^*(v)\big)\\ &= (\pi_1 \circ \Sigma\Psi)^*\iota_1^*(v) + (\pi_2 \circ \Sigma\Psi)^*\iota_2^*(v) = \iota_1^*(v) + \iota_2^*(v).\end{aligned}$$

Here, we have used the fact that $\pi_i \circ \Sigma\Psi$ is homotopic to Id_{S^k} (for $i = 1$ or 2).

Now suppose that we have two clutching functions $\varphi_1, \varphi_2 : S^{k-1} \to \mathrm{SO}(n)$, both of which send the basepoint of S^{k-1} to the identity matrix $\mathrm{Id} \in \mathrm{SO}(n)$. Let ξ_1 and ξ_2 be the vector bundles over S^k defined by the clutching functions φ_1 and φ_2, respectively. We also define a bundle $\xi_1 \vee \xi_2$ over $S^k \vee S^k$ by gluing the fibers of ξ_1 and ξ_2 over the basepoint of S^k.[9] Then it is clear that

$$\iota_1^*(\xi_1 \vee \xi_2) = \xi_1 \qquad \text{and} \qquad \iota_2^*(\xi_1 \vee \xi_2) = \xi_2.$$

But we get the same vector bundle if we first take a wedge sum and then perform the clutching construction,[10] so $\xi_1 \vee \xi_2$ is described by the clutching function

$$\varphi_1 \vee \varphi_2 : S^{k-1} \vee S^{k-1} \to \mathrm{SO}(n).$$

In $\pi_{k-1}\mathrm{SO}(n)$, the sum of φ_1 and φ_2 is represented by the map $(\varphi_1 \vee \varphi_2) \circ \Psi$; we let ξ denote the vector bundle over S^k defined by this clutching function. By the naturality property stated at the end of §2.1, we get $\xi \cong \Sigma\Psi^*(\xi_1 \vee \xi_2)$. We can now conclude the desired homomorphism property:

$$\begin{aligned}c(\xi) &= c\big(\Sigma\Psi^*(\xi_1 \vee \xi_2)\big) = \Sigma\Psi^* c(\xi_1 \vee \xi_2)\\ &= \iota_1^* c(\xi_1 \vee \xi_2) + \iota_2^* c(\xi_1 \vee \xi_2)\\ &= c\big(\iota_1^*(\xi_1 \vee \xi_2)\big) + c\big(\iota_2^*(\xi_1 \vee \xi_2)\big) = c(\xi_1) + c(\xi_2).\end{aligned}$$

2.4 Oriented S^3-bundles over S^4

We now know enough about the clutching construction to focus in on the case that is of interest to us: oriented $\mathbb{R}^4$-bundles over S^4. Their unit sphere bundles are oriented 7-manifolds, which we will take as candidates for exotic 7-spheres.

[9] Because φ_i takes the basepoint of S^{k-1} to $\mathrm{Id} \in \mathrm{SO}(n)$, the clutching construction yields a canonical identification of $\mathbb{R}^n$ with the fiber of ξ_i over the basepoint of S^k (for $i = 1$ or 2). It is not hard to check local trivializability of $\xi_1 \vee \xi_2$, by gluing together local trivializations of the bundles ξ_1 and ξ_2 near the basepoint of S^k.

[10] In short, this follows from the fact that the reduced cone functor and wedge sum commute, so the two gluings used to define $\xi_1 \vee \xi_2$ are essentially the same in either order.

Consider the mapping $\Phi : S^3 \times S^3 \to \mathrm{SO}(4)$ defined by $\Phi(x,y) : v \mapsto xvy$, where we identify $\mathbb{R}^4 = \mathbb{H}$ and view $S^3 \subseteq \mathbb{H}$ as the group of unit quaternions. This induces a homomorphism[11]

$$\mathbb{Z} \oplus \mathbb{Z} \cong \pi_3(S^3 \times S^3) \xrightarrow{\quad \Phi_* \quad} \pi_3\mathrm{SO}(4).$$

For any $(h,\ell) \in \mathbb{Z}\oplus\mathbb{Z}$, the corresponding element of $\pi_3(S^3\times S^3)$ is the homotopy class of the map $S^3 \to S^3 \times S^3$ defined by $u \mapsto (u^h, u^\ell)$. The homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \pi_3\mathrm{SO}(4)$ therefore sends any pair $(h,\ell) \in \mathbb{Z} \oplus \mathbb{Z}$ to the homotopy class of the map $\varphi_{h,\ell} : S^3 \to \mathrm{SO}(4)$ given by

$$\varphi_{h,\ell}(u) : v \mapsto u^h v u^\ell$$

This clutching function defines an oriented $\mathbb{R}^4$-bundle $\xi_{h,\ell}$ over S^4. Notice that

$$\overline{\varphi_{h,\ell}(u)\bar{v}} = \overline{u^h \bar{v} u^\ell} = \bar{u}^\ell v \bar{u}^h = u^{-\ell} v u^{-h} = \varphi_{-\ell,-h}(u)v.$$

Because the transformation $T \in \mathrm{O}(4)$ given by $T(v) = \bar{v}$ is orientation-reversing, the results of §2.1 show that the two clutching functions $\varphi_{h,\ell}$ and $\varphi_{-\ell,-h}$ define the same bundle, but with opposite orientations. It follows immediately that

$$p_1(\xi_{h,\ell}) = p_1(\xi_{-\ell,-h}).$$

From §2.3, we know that the map $(h,\ell) \mapsto p_1(\xi_{h,\ell})$, given by the composition

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_3\mathrm{SO}(4) \longrightarrow \left\{ \begin{array}{c} \text{isomorphism} \\ \text{classes of oriented} \\ \mathbb{R}^4\text{-bundles over } S^4 \end{array} \right\} \xrightarrow{p_1} H_4(S^4),$$

is a homomorphism. Taking β to be a generator of $H^4(S^4) \cong \mathbb{Z}$, we then have

$$p_1(\xi_{h,\ell}) = (ah + b\ell)\beta$$

for some $a, b \in \mathbb{Z}$. We can immediately see that $-b = a$, because

$$a\beta = p_1(\xi_{1,0}) = p_1(\xi_{0,-1}) = -b\beta.$$

Moreover, we calculated in §2.2 that the bundle $\gamma_2^{\mathbb{H}}$ is described by the clutching function $\varphi_{1,0}$. By the calculation from §1.2, it follows that $a = \pm 2$, because

$$a\beta = p_1(\xi_{1,0}) = p_1(\gamma_2^{\mathbb{H}}) = \pm 2\beta.$$

Therefore, we have calculated[12] the first Pontryagin class $p_1(\xi_{h,\ell}) = \pm 2(h-\ell)\beta$.

The sphere bundles $S(\xi_{h,\ell})$ are oriented 7-manifolds, which we take as our candidates for exotic 7-spheres. In the following two sections, we will determine some conditions on $(h,\ell) \in \mathbb{Z}\oplus\mathbb{Z}$ that determine when $S(\xi_{h,\ell})$ is homeomorphic, but not diffeomorphic to S^7. In particular, we will use the calculation of $p_1(\xi_{h,\ell})$ to identify situations in which $S(\xi_{h,\ell})$ and S^7 cannot be diffeomorphic.

[11] In fact, this is an isomorphism, because Φ is a double cover. See §5.1 for further details.

[12] A similar calculation can be used to determine the Euler class $e(\xi_{h,\ell}) = \pm(h+\ell)$.

3 Not Diffeomorphic...

3.1 The signature formula

Let X be a smooth, closed, oriented 8-manifold. Then the composition

$$H^4(X;\mathbb{R}) \times H^4(X;\mathbb{R}) \xrightarrow{\quad\smile\quad} H^8(X;\mathbb{R}) \xrightarrow{\frown [X]} \mathbb{R}$$

is a symmetric bilinear form on the finite-dimensional vector space $H^4(X;\mathbb{R})$, so it can be diagonalized. In terms of this form, the signature of X is defined as

$$\sigma(X) = \#\{\text{positive eigenvalues}\} - \#\{\text{negative eigenvalues}\}.$$

This number is a homotopy invariant (it only depends on the cohomology ring), but Thom observed[13] a fundamental relationship with certain smooth invariants, namely the Pontryagin numbers:

$$\sigma(X) = \frac{1}{45}\big\langle 7p_2(TX) - p_1(TX)^2, [X]\big\rangle$$

Rearranging, we can see that

$$\big\langle p_1(TX)^2, [X]\big\rangle = 7\big\langle p_2(TX), [X]\big\rangle - 45\sigma(X)$$

and hence $\big\langle p_1(TX)^2, [X]\big\rangle \equiv 4\sigma(X) \pmod 7$. Now suppose that $S(\xi_{h,\ell})$ and S^7 are diffeomorphic. Then we can define a smooth, closed, oriented 8-manifold

$$X = D(\xi_{h,\ell}) \cup_{S^7} D^8,$$

where we glue the disk D^8 to the disk bundle $D(\xi_{h,\ell})$ by a smooth identification of their boundaries $S^7 \cong S(\xi_{h,\ell})$. With the right orientation, we will show that

$$\sigma(X) = 1 \qquad \text{and} \qquad \big\langle p_1(TX)^2, [X]\big\rangle = 4(h-\ell)^2.$$

This will imply that $4(h-\ell)^2 \equiv 4 \pmod 7$ and therefore $(h-\ell)^2 \equiv 1 \pmod 7$. For these calculations, we will abbreviate $S = S(\xi_{h,\ell}) \cong S^7$ and $D = D(\xi_{h,\ell})$.

3.2 Calculating $\sigma(X)$

Since D^8 is a cone on S^7, we can view the manifold X as the mapping cone for the inclusion $S \hookrightarrow D$, so we obtain a ring isomorphism $H^*(D,S) \cong \widetilde{H}^*(X)$. Writing u for the Thom class of $\xi_{h,\ell}$, we now consider the following diagram:

[13] This result is called the Hirzebruch signature formula, as it was generalized by Hirzebruch to an explicit formula in any dimension $4k$ (where $k = 2$ in our case).

$$
\begin{array}{ccccc}
 & & H^3(S) = 0 & & \\
 & & \downarrow & & \\
\mathbb{Z} = H^0(D) & \xrightarrow{\smile u} & H^4(D,S) & & \\
 & & \downarrow & & \\
 & & H^4(D) & \xrightarrow{\smile u} & H^8(D,S) \\
 & & \downarrow & & \\
 & & H^4(S) = 0 & &
\end{array}
\qquad H^4(D,S) \xrightarrow{\smile u} H^8(D,S)
$$

The horizontal maps are isomorphisms by the Thom isomorphism theorem, while the vertical maps come from the long exact sequence of the pair (D, S). Since $S \cong S^7$ has trivial cohomology in degrees 3 and 4, the middle vertical map is an isomorphism. We therefore have

$$\mathbb{Z} = H^0(D) \cong H^4(D,S) \cong H^4(D) \cong H^8(D,S),$$

with generators $u \in H^4(D,S)$ and $u^2 \in H^8(D,S)$. Correspondingly, we have

$$\mathbb{Z} \cong H^4(X) \cong H^8(X),$$

with some generators $\alpha \in H^4(X)$ and $\alpha^2 \in H^8(X)$. Choosing the appropriate orientation on X, we then have $\left\langle \alpha^2, [X] \right\rangle = 1$ and thus $\sigma(X) = 1$.

3.3 Calculating $p_1(TX)$

Applying the Mayer-Vietoris sequence to the decomposition $X = D \cup_{S^7} D^8$ gives

$$0 = H^3(S^7) \longrightarrow H^4(X) \longrightarrow H^4(D) \oplus H^4(D^8) \longrightarrow H^4(S^7) = 0$$

Because $H^4(D^8) = 0$, exactness implies that the inclusion $i : D \hookrightarrow X$ induces an isomorphism on H^4. The zero section $s : S^4 \hookrightarrow D$ is a homotopy equivalence, so it also induces an isomorphism on H^4. Therefore, we have

$$H^4(X) \cong H^4(D) \cong H^4(S^4) \cong \mathbb{Z}.$$

We will prove that $s^* i^* p_1(TX) = p_1(s^* i^* TX) = \pm 2(h - \ell)\beta$, where $\beta \in H^4(S^4)$ is a generator. It then follows that $p_1(TX) = \pm 2(h - \ell)\alpha$ and therefore that

$$\langle p_1(TX)^2, [X] \rangle = \left\langle 4(h-\ell)^2 \alpha^2, [X] \right\rangle = 4(h-\ell)^2.$$

Since D is a submanifold of X having full dimension, we have $i^*TX = TD$. Identifying S^4 with its image under the zero section s, we have a splitting

$$s^*TD \cong TS^4 \oplus \xi_{h,\ell}$$

into vectors tangent to the base and vectors tangent to the fiber (this applies to any smooth vector bundle). Because Pontryagin classes obey a Whitney sum formula up to 2-torsion and $H^4(S^4) \cong \mathbb{Z}$ is torsion-free, we now have

$$\begin{aligned} p_1(s^*i^*TX) &= p_1(TS^4 \oplus \xi_{h,\ell}) \\ &= p_0(TS^4) \smile p_1(\xi_{h,\ell}) + p_1(TS^4) \smile p_0(\xi_{h,\ell}) \\ &= p_1(\xi_{h,\ell}) + p_1(TS^4) = \pm 2(h-\ell)\beta + p_1(TS^4) \end{aligned}$$

In the last equality, we have used the calculation from §2.4. Pontryagin numbers are an oriented cobordism invariant, so the fact that $\partial D^5 = S^4$ implies that

$$\left\langle p_1(TS^4), [S^4] \right\rangle = 0$$

and thus $p_1(TS^4) = 0$. This completes the promised computations and shows that the existence of a diffeomorphism $S(\xi_{h,\ell}) \cong S^7$ implies $(h-\ell)^2 \equiv 1 \pmod 7$. We will employ the contrapositive result to detect when no such diffeomorphism between $S(\xi_{h,\ell})$ and S^7 can exist.[14]

4 . . . But Homeomorphic

4.1 Detecting topological spheres

We will now show that $h + \ell = 1$ implies that $S(\xi_{h,\ell})$ is homeomorphic to S^7. The proof relies on Morse theory, which (classically) investigates the structure of manifolds via smooth functions having only non-degenerate critical points. A quintessential example[15] is the norm-squared function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = |x|^2 = x_1^2 + \cdots + x_n^2.$$

This has a unique critical point at the origin, which is non-degenerate.

[14] At this point, the reader may wonder why we had to go through an analysis of the tangent bundle to an auxiliary 8-manifold X, to tell apart the smooth structures on $S(\xi_{h,\ell})$ and S^7. Why not look directly at the tangent bundles of these 7-manifolds? It turns out that every $\mathbb{R}^7$-bundle over S^7 is trivial (see this MathOverflow answer from Ian Agol), so characteristic classes on the tangent bundle are insufficient to tell apart smooth structures on S^7. However, it is possible to tell apart certain closed 7-manifolds by comparing properties of 8-manifolds that they bound. See [M1] for details.

[15] Indeed, if we allow each summand x_i^2 to have a sign of ± 1, then the Morse lemma states that *every* non-degenerate critical point has this form in some chart. For a proof, see [M2].

The Morse-theoretic result that we will use is called the Reeb sphere theorem:

Theorem. Suppose that M is a closed n-manifold and $f : M \to \mathbb{R}$ is a smooth function having exactly two critical points, both of which are non-degenerate.[16] Then M is homeomorphic to S^n.

As a particularly simple example of this phenomenon, consider a non-zero linear functional $\lambda : \mathbb{R}^{n+1} \to \mathbb{R}$. Restricting to the unit sphere, we get a smooth map $\lambda : S^n \to \mathbb{R}$ with exactly two critical points, both of which are non-degenerate (explicitly, these are maximum and minimum of λ, the points in $(\ker \lambda)^\perp \cap S^n$). While S^n is obviously diffeomorphic to itself, this theorem can only guarantee a homeomorphism, as we will see when we use it to detect exotic spheres.

4.2 Defining a useful function

To apply this theorem to $S(\xi_{h,\ell})$, we will first recall the construction of $\xi_{h,\ell}$. This vector bundle is defined by the clutching function $\varphi_{h,\ell} : S^3 \to \mathrm{SO}(4)$ given by $\varphi_{h,\ell}(u) : v \mapsto u^h v u^\ell$. This means that its total space is defined as a gluing

$$E(\xi_{h,\ell}) = (\mathbb{R}^4_+ \times \mathbb{R}_4) \cup (\mathbb{R}^4_- \times \mathbb{R}^4),$$

where the gluing map associates $(u, v) \in \mathbb{R}^4_+ \backslash 0 \times \mathbb{R}^4$ and $(u', v') \in \mathbb{R}^4_- \backslash 0 \times \mathbb{R}^4$ by

$$(u', v') = \left(\frac{u}{|u|^2}, \frac{u^h}{|u|^h} v \frac{u^\ell}{|u|^\ell} \right) = \left(\frac{u}{|u|^2}, u^h v u^{-h} \frac{u}{|u|} \right).$$

In the last equality above, we used the assumption that $h + \ell = 1$. Now we have

$$S(\xi_{h,\ell}) = (\mathbb{R}^4_+ \times S^3) \cup (\mathbb{R}^4_- \times S^3),$$

defined by the restriction of the same gluing map. We henceforth focus on this sphere bundle, so it will be assumed that $v, v' \in S^3$. Since $v' \in S^3$ is invertible, we can reparametrize $\mathbb{R}^4_- \times S^3$ by mapping $(u', v') \mapsto (u'', v')$ with $u'' = u'(v')^{-1}$. Whenever $u \neq 0$, the point (u'', v') corresponding to (u, v) is described by

$$u'' = u'(v')^{-1} = \frac{u}{|u|^2} \frac{u^{-1}}{|u|^{-1}} u^h v^{-1} u^{-h} = \frac{u^h v^{-1} u^{-h}}{|u|}$$

In particular, whenever $u \neq 0$, we have $|u''| = |u|^{-1}$ and

$$\mathrm{Re}(u'') = \frac{\mathrm{Re}(v^{-1})}{|u|} = \frac{\mathrm{Re}(\bar{v})}{|u|} = \frac{\mathrm{Re}(v)}{|u|},$$

[16] The non-degeneracy condition is not actually necessary, but it makes the proof simpler. For proofs of both versions of the theorem, see [M3].

because $|v| = 1$ and the real part of a quaternion is fixed under conjugation.[17]

We define a smooth function $f : S(\xi_{h,\ell}) \to \mathbb{R}$ by

$$f(u, v) = \frac{\mathrm{Re}(v)}{\sqrt{1 + |u|^2}} \qquad \text{and} \qquad f(u'', v') = \frac{\mathrm{Re}(u'')}{\sqrt{1 + |u''|^2}}$$

Using the above computations, it is straightforward to check that these formulas agree on the intersection of their domains. We will show that f has exactly two critical points, both of which are non-degenerate.

4.3 Calculating critical points

To find the critical points of f, first let $u'' = a + bi + cj + dk$ and notice that

$$f(u'', v') = \frac{a}{\sqrt{1 + a^2 + b^2 + c^2 + d^2}} = \frac{a}{\sqrt{e + a^2}}$$

where $e \geq 1$ is constant as a function of a. A straightfoward computation shows that the partial derivative with respect to a is always positive. Hence, there are no critical points on $\mathbb{R}^4_- \times S^3$ and it remains to check points $(u, v) \in \mathbb{R}^4_+ \times S^3$ such that $u = 0$. For this calculation, we will use the following lemma:

Lemma. Let M and N be smooth manifolds equipped with smooth functions $g_1 : M \to \mathbb{R}$ and $g_2 : N \to \mathbb{R}$. If $x \in M$ is a critical point of g_1 with $g_1(x) \neq 0$, then a point $y \in N$ is a critical point of g_2 if and only if (x, y) is a critical point of the multiplied function $g_1 g_2 : M \times N \to \mathbb{R}$. When this is the case, the critical point (x, y) is non-degenerate if and only if both x and y are non-degenerate.

Proof. Since we are only checking local conditions, we may assume that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. We assumed that $\nabla g_1(x) = 0$, so the Leibniz rule gives

$$\nabla(g_1 g_2)(x, y) = g_1(x)\nabla g_2(y) + g_2(y)\nabla g_1(x) = g_1(x)\nabla g_2(y).$$

But we also assumed that $g_1(x) \neq 0$, so we can see that

$$\nabla(g_1 g_2)(x, y) = 0 \iff \nabla g_2(y) = 0.$$

[17] To highlight the relevance of conjugation, we compare two clutching functions: our $\varphi_{h,\ell}$ (with $h + \ell = 1$) and the function $\varphi_{1,0}$ that was used to define $\gamma_2^{\mathbb{H}}$. These two are related by

$$\varphi_{h,\ell}(u)v = u^h v u^{1-h} = u^{h-1}(uv)u^{1-h} = u^{h-1}\big(\varphi_{1,0}(u)v\big)u^{1-h}.$$

This suggests a heuristic view of $S(\xi_{h,\ell})$ (with $h + \ell = 1$) as a variant of the Hopf bundle

$$S^3 \hookrightarrow S^7 \to S^4$$

that has been "twisted by conjugation." This highlights the importance of non-commutativity in this construction and provides an *a posteriori* justification of our choice to set $h + \ell = 1$ when looking for exotic 7-spheres. For the *a priori* reasoning that led Milnor to set $h + \ell = 1$, see the calculation carried out in §5.2.

Writing $\mathbf{H}_p(g)$ for the Hessian of a function g at a point p, we also have

$$\mathbf{H}_{(x,y)}(g_1 g_2) = \begin{bmatrix} \mathbf{H}_x(g_1) & \nabla g_1(x) \nabla g_2(y)^T \\ \nabla g_2(y) \nabla g_1(x)^T & \mathbf{H}_y(g_2) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_x(g_1) & 0 \\ 0 & \mathbf{H}_y(g_2) \end{bmatrix}$$

This matrix is invertible if and only if $\mathbf{H}_x(g_1)$ and $\mathbf{H}_y(g_2)$ are invertible.[18] □

A brief calculation shows that we have a diffeomorphism $(-1, \infty) \to (0, \infty)$ given by $1/\sqrt{1+t}$. Composing this with the first example of §4.1, the function

$$g_1(u) = \frac{1}{\sqrt{1 + |u|^2}}$$

has a non-degenerate critical point at $u = 0$. The second example of §4.1 shows that $\text{Re} : S^3 \to \mathbb{R}$ has critical points at $v = \pm 1$, which are both non-degenerate. By the lemma, we see that $f(u, v) = g_1(u)\text{Re}(v)$ has exactly two critical points, both of which are non-degenerate. Using the theorem from §4.1, we can finally see that $S(\xi_{h,\ell})$ is homeomorphic to S^7.

Combining the results of §3 and §4, we have now shown the following:

$$\begin{array}{ccc} (h-\ell)^2 \equiv 1 \pmod 7 & \Longleftarrow & S(\xi_{h,\ell}) \text{ is diffeomorphic to } S^7 \\ & & \Big\Downarrow \\ h + \ell = 1 & \Longrightarrow & S(\xi_{h,\ell}) \text{ is homeomorphic to } S^7 \end{array}$$

As such, the smooth manifold $S(\xi_{h,1-h})$ is homeomorphic to S^7 for any $h \in \mathbb{Z}$, but it can only be diffeomorphic to S^7 when $(2h-1)^2 \equiv 1 \pmod 7$. Therefore, we get an exotic smooth structure on S^7 from any integer $h \not\equiv 0, 1 \pmod 7$.[19]

5 Appendices

5.1 Rotations in $\mathbb{R}^4$ via quaternions

In this appendix, we will prove a fact mentioned in §2.4, namely that the map

$$\Phi : S^3 \times S^3 \to \text{SO}(4) \quad \text{given by} \quad \Phi(x, y) : v \mapsto xvy$$

[18] You may notice that the computation of the Hessian did not depend on y being a critical point of g_2. But the Hessian is only defined in a *coordinate-invariant* way at critical points.

[19] Using only the techniques developed here, we have no way of telling apart exotic spheres. Up to orientation-preserving diffeomorphism, it turns out that there are 28 oriented smooth structures on S^7, but only 16 can be produced as a bundle in the fashion outlined here [EK].

is a double cover. It is clearly equivalent to prove that

$$\Phi' : S^3 \times S^3 \to \mathrm{SO}(4) \quad \text{given by} \quad \Phi'(x,y) : v \mapsto xvy^{-1}$$

is a double cover, since inversion in S^3 is a diffeomorphism. To prove this fact, note that Φ' is a smooth homomorphism between compact, connected Lie groups of the same dimension. To show that Φ' is a double cover, it therefore suffices to show that $\ker \Phi'$ contains exactly two elements.

Let us first calculate the center $Z(\mathbb{H})$. If $u = a + bi + cj + dk$, then we have

$$\begin{aligned} iui^{-1} &= a + bi - cj - dk, \\ juj^{-1} &= a - bi + cj - dk, \\ kuk^{-1} &= a - bi - cj + dk. \end{aligned}$$

So if $u \in Z(\mathbb{H})$, then we must have $b = c = d = 0$. This proves that $Z(\mathbb{H}) \subseteq \mathbb{R}$. The reverse inclusion is apparent, so we in fact have $Z(\mathbb{H}) = \mathbb{R}$.

Now suppose that $\Phi'(x,y) = \mathrm{Id}$, i.e. $x, y \in S^3$ and $xvy^{-1} = v$ for all $v \in \mathbb{H}$. This implies that $xy^{-1} = 1$ and thus $x = y$, so we are analyzing the map

$$\Phi'(x,y) : v \mapsto xvx^{-1}$$

This is the identity if and only if $x \in Z(\mathbb{H}) = \mathbb{R}$. Since x is in the unit sphere, we must have $x = \pm 1$. This proves that $\ker \Phi' = \big\{(1,1), (-1,-1)\big\}$ and hence that Φ' is a double cover.

We have concretely described rotations in $\mathbb{R}^4$ via quaternionic multiplication. This also restricts to a description of rotations in $\mathbb{R}^3$. We view $\mathrm{SO}(3) \subseteq \mathrm{SO}(4)$ as the set of rotations of that restrict to the identity on $\mathbb{R} \subseteq \mathbb{H}$, since these are described by rotations of the 3-dimensional space of imaginary quaternions:

$$\mathbb{R}^\perp = \{bi + cj + dk : b, c, d \in \mathbb{R}\}.$$

Similarly to the above analysis of SO(4), we can see that

$$\Phi'(x,y) \in \mathrm{SO}(3) \iff \Phi'(x,y) \text{ fixes } 1 \iff x = y.$$

Thus Φ' restricts to a double cover of SO(3) by the diagonal $\Delta \subseteq S^3 \times S^3$:

$$\Phi'|_\Delta : S^3 \to \mathrm{SO}(3) \quad \text{given by} \quad \Phi'|_\Delta(x) : v \mapsto xvx^{-1}.$$

To conclude this section, we will also show that the n^{th} power map $S^3 \to S^3$ has degree n, since this fact was implicitly invoked in §2.4.[20] For this purpose, we consider the following two maps:

[20] One may be tempted to calculate the degree as a signed count of points in a regular fiber, mimicking the analogous result for $S^1 \subseteq \mathbb{C}$. This is probably doable, but non-commutativity takes some elegance out of this approach: the n^{th} power map is not a local diffeomorphism when $n \geq 2$ (for instance, notice that *any* purely imaginary $u \in S^3$ squares to -1). Instead, we use the homological definition of degree to give a fairly succinct argument.

$$\begin{array}{ccccc} S^3 & \xrightarrow{\quad\Delta\quad} & (S^3)^n & \xrightarrow{\quad\Pi\quad} & S^3 \\ x & \longmapsto & (x,\dots,x) \qquad (x_1,\dots,x_n) & \longmapsto & \prod_{i=1}^n x_i \end{array}$$

Then $\Pi \circ \Delta(x) = x^n$ is our n^{th} power map. Applying the homology functor H_3 and the Künneth theorem, we get the following diagram of homomorphisms:

$$\begin{array}{ccccc} H_3(S^3) & \xrightarrow{\Delta_*} & H_3\big((S^3)^n\big) & \xrightarrow{\Pi_*} & H_3(S^3) \\ \| & & \| & & \| \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z} \end{array}$$

Let $\pi_i : (S^3)^n \to S^3$ denote the projection to i^{th} factor and let $\iota_i : S^3 \to (S^3)^n$ denote inclusion of the i^{th} factor (where the other factors map constantly to 1). The isomorphism $H_3\big((S^3))^n\big) \cong \mathbb{Z}^n$ can be described either as a direct product via the projections $(\pi_i)_*$ or as a direct sum via the inclusions $(\iota_i)_*$. Notice that

$$\pi_i \circ \Delta = \mathrm{Id}_{S^3} \text{ and } \Pi \circ \iota_i = \mathrm{Id}_{S^3}.$$

From this, we can see that our induced maps on H_3 take the following form:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\quad\Delta_*\quad} & \mathbb{Z}^n & \xrightarrow{\quad\Pi_*\quad} & \mathbb{Z} \\ a & \longmapsto & (a,\dots,a) \qquad (a_1,\dots,a_n) & \longmapsto & \sum_{i=1}^n a_i \end{array}$$

Finally, we can see that $\Pi_* \circ \Delta_*(a) = na$ and therefore that $\Pi \circ \Delta$ has degree n.

5.2 Why did we assume $h + \ell = 1$?

We know that $S(\xi_{h,\ell})$ is homeomorphic to S^7 whenever $h + \ell = 1$, but we still may wonder why Milnor knew to investigate these particular values of h and ℓ. To explain his process, we will show that $S(\xi_{h,\ell})$ is homotopy equivalent to S^7 if and only if $h + \ell = \pm 1$.[21] Since $S(\xi_{h,\ell})$ and $S(\xi_{-\ell,-h})$ are diffeomorphic with opposite orientation, this reasonably justifies focusing on the case of $h + \ell = 1$.

Recall that $S(\xi_{h,\ell})$ is an S^3-bundle over S^4. From the corresponding long exact sequence of homotopy groups, we can see that $\pi_i S(\xi_{h,\ell})$ vanishes for $i \leq 2$. Thus $S(\xi_{h,\ell})$ is a homotopy 7-sphere if and only if we have

$$H_i S(\xi_{h,\ell}) \cong \begin{cases} 0, & i \neq 7 \\ \mathbb{Z}, & i = 7 \end{cases} \text{ for all } i \geq 3.$$

[21] This calculation is much more standard than the use of Morse functions, and was carried out prior to Milnor's realization that he had discovered exotic smooth structures on S^7 [M5].

By restricting the gluing used to define $E(\xi_{h,\ell})$, we can write

$$S(\xi_{h,\ell}) = (D^4_+ \times S^3) \cup (D^4_- \times S^3),$$

where the gluing is given by the following pushout diagram:

$$\begin{array}{ccccc}
(u,v) & \longmapsto & & & \big(u, u^h v u^\ell\big) \\
\big\downarrow & & S^3 \times S^3 & \hookrightarrow & D^3_- \times S^3 \\
 & & \big\downarrow & & \big\downarrow \\
(u,v) & & D^3_+ \times S^3 & \hookrightarrow & S(\xi_{h,\ell})
\end{array}$$

Applying the homology functor H_3 to this diagram, we ignore the contractible factors $D^3_\pm$ and just look at the degrees in each component of the maps

$$(u,v) \mapsto v \qquad \text{and} \qquad (u,v) \mapsto u^h v u^\ell.$$

We can see that the map $(u,v) \mapsto u^h v u^\ell$ has degree $h + \ell$ as a function of u and degree 1 as a function of v (letting the other vector equal $1 \in S^3$ in each case). So after applying H_3, we have the following diagram:

$$\begin{array}{ccc}
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\begin{bmatrix} h+\ell & 1 \end{bmatrix}} & \mathbb{Z} \\
{\scriptstyle \begin{bmatrix} 0 & 1 \end{bmatrix}} \big\downarrow & & \big\downarrow \\
\mathbb{Z} & \longrightarrow & H_3 S(\xi_{h,\ell})
\end{array}$$

This is no longer a pushout diagram; instead, we have a Mayer-Vietoris sequence, which includes the following portion:

$$0 \longrightarrow H_4 S(\xi_{h,\ell}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 0 & 1 \\ h+\ell & 1 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_3 S(\xi_{h,\ell}) \longrightarrow 0$$

By exactness, we can see that $H_3 S(\xi_{h,\ell})$ and $H_4 S(\xi_{h,\ell})$ both vanish if and only if the middle map is an isomorphism. This holds precisely when the determinant is invertible in $\mathbb{Z}$, i.e. when $h + \ell = \pm 1$. Regardless of h and ℓ, given any $i \geq 5$, we can use another portion of the same Mayer-Vietoris sequence:

$$0 \longrightarrow H_i S(\xi_{h,\ell}) \longrightarrow H_{i-1}(S^3 \times S^3) \longrightarrow 0$$

For any $i \geq 5$, by exactness and the Künneth theorem, we then have:

$$H_i S(\xi_{h,\ell}) \cong H_{i-1}(S^3 \times S^3) \cong \begin{cases} 0, & i \neq 7 \\ \mathbb{Z}, & i = 7 \end{cases}$$

Therefore $S(\xi_{h,\ell})$ is homotopy equivalent to S^7 if and only if $h + \ell = \pm 1$.

5.3 A fun fact about $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$

We conclude with an unrelated fun fact about the complex vector bundle $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$. Given any real vector bundle η, the complexification $\xi = \eta \otimes \mathbb{C}$ satisfies $\bar{\xi} \cong \xi$. This implies that $2c_i(\xi) = 0$ whenever i is odd, justifying our ignorance of odd Chern classes in the definition of Pontryagin classes. But the lovely text [FF] erroneously claims the converse, namely that an isomorphism $\bar{\xi} \cong \xi$ implies that ξ is the complexification of some real vector bundle η. This greatly confused me when I was first learning about characteristic classes, but I eventually learned from Alexander Givental and Tong Zhou that $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$ is a counterexample.

Suppose $\gamma_2^{\mathbb{H}}|_{\mathbb{C}} \cong \eta \otimes \mathbb{C}$ for some $\mathbb{R}^2$-bundle η over $\mathbb{HP}^1$. We have $w_1(\eta) = 0$ because $H^1(\mathbb{HP}^1; \mathbb{Z}/2)$ is trivial, so η must be orientable. This yields an integral Euler class $e(\eta) = 0$, since $H^2(\mathbb{HP}^1; \mathbb{Z})$ is also trivial. But we also know that

$$\begin{aligned} e(\eta) = 0 &\iff \eta \text{ has a non-vanishing section} \\ &\implies \gamma_2^{\mathbb{H}}|_{\mathbb{C}} \cong \eta \otimes \mathbb{C} \text{ has a non-vanishing section.} \end{aligned}$$

This implies that $\gamma_2^{\mathbb{H}}$ is a quaternionic line bundle with a non-vanishing section, so it is trivial. But then $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$ is trivial, which contradicts §1.2, where we showed

$$c_2(\gamma_2^{\mathbb{H}}|_{\mathbb{C}}) \neq 0.$$

Therefore, no such η can exist. It now remains to show that $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$ is isomorphic to its conjugate, or equivalently, that there is a conjugate-linear automorphism of the $\mathbb{C}^2$-bundle $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$. Since $\mathbb{HP}^1$ consists of right-lines and our total space is

$$E(\gamma_2^{\mathbb{H}}) = \{(v, \ell) \in \mathbb{H}^2 \times \mathbb{HP}^1 : v \in \ell\},$$

it follows that the map $(v, \ell) \mapsto (vj, \ell)$ is an automorphism of $\gamma_2^{\mathbb{H}}|_{\mathbb{R}}$. To see this as a conjugate-linear automorphism of $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$, recall that the complex structure on $\gamma_2^{\mathbb{H}}|_{\mathbb{C}}$ comes from right-multiplication by i. Now since i and j anti-commute, right-multiplication by j is conjugate-linear (over $\mathbb{C}$) on each fiber $\ell \in \mathbb{HP}^1$.

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