The Generalized Schoenflies Theorem

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In this note, I will spell out various details of the proofs in [B]. Remarkably, the techniques used in that paper are all of the sort that one would encounter in a first course on point-set topology (e.g. Chapters 2-4 of [M]). I have done my best to gear my explanations towards a reader who only has this background, although we will need to black-box one result, which is proved using homology.

Contents

1 Separating Spheres

We first establish some notation and terminology. Let S^n denote the n-sphere, let $\bar{D}^n$ denote the closed n-disk, let D^n denote its interior and $\partial D^n = \bar{D}^n - D^n$. We will write "map" to mean "continuous function." Given any map $f : X \to Y$, an inverse set under f is any fiber $f^{-1}(y) \subseteq X$ containing more than one point. We will assume that $n > 1$, so that S^{n-1} is connected (most of the results, including Theorems 1.1 and 5.1, are still true and much more straightforward to prove when $n = 0$ or 1).

Our first result is the only one that we have to state without a full proof:

Theorem 1.1 (Jordan-Brouwer Theorem)**.** *For any injective map $f : \bar{D}^k \to S^n$, the complement $S^n - f(\bar{D}^k)$ is connected. For any injective map $g : S^{n-1} \to S^n$, the complement $S^n - g(S^{n-1})$ has two components, each with boundary $g(S^{n-1})$.*

Proof. Homology theory is required to prove the connectedness of $S^n - f(\bar{D}^k)$ and the fact that $S^n - g(S^{n-1})$ has two components (Proposition 2B.1 in [H]). As such, we will take this result for granted and move on to recounting the proof of the final claim, as it is laid out in [S].

Let B and C be the two components of $S^n - g(S^{n-1})$. Connected components of any topological space are closed, so B and C are both closed in $S^n - g(S^{n-1})$. But they are each others' complements, so they are both open in $S^n - g(S^{n-1})$. Since $g(S^{n-1})$ is compact and thus closed in S^n, it follows that $S^n - g(S^{n-1})$ is open in S^n. Thus B and C are open in S^n. We will show that $\partial B = g(S^{n-1})$ (an analogous proof applies to C). Note that we have a disjoint union

$$S^n = B \sqcup C \sqcup g(S^{n-1}).$$

Since B is open, we have $\partial B = \bar{B} - B$, so we need to prove $\bar{B} = B \sqcup g(S^{n-1})$. But C is also open, so $B \sqcup g(S^{n-1})$ is closed and thus $B \subseteq \bar{B} \subseteq B \sqcup g(S^{n-1})$. For the sake of contradiction, suppose that there exists some $x \in g(S^{n-1}) - \bar{B}$. Since $x \notin \bar{B}$, there is an open set $U \subseteq S^n$ with $x \in U$ and $U \cap B = \emptyset$.

Let $\tilde{x} = g^{-1}(x)$ and consider the open neighborhood $g^{-1}(U) \ni \tilde{x}$ in S^{n-1}. Letting $V \ni \tilde{x}$ be a standard open disk,[1] which is small enough that $V \subseteq g^{-1}(U)$, we can see that the complement $E = S^{n-1} - V$ is homeomorphic to $\bar{D}^{n-1}$. Thus

$$S^n - g(E) = B \sqcup C \sqcup g(V)$$

is connected, by the first portion of this theorem. The open set $W = U - g(E)$ must satisfy $W \subseteq C \sqcup g(V)$, because $W \subseteq S^n - g(E)$ and $W \cap B \subseteq U \cap B = \emptyset$. We also get $g(V) \subseteq W$, because $g(V) \subseteq U$ by assumption and $g(V) \cap g(E) = \emptyset$ by the injectivity of g. Thus $C \sqcup g(V) = C \cup W$ and so we can write

$$S^n - g(E) = B \sqcup C \sqcup g(V) = B \sqcup (C \cup W).$$

Since B, C and W are open, this contradicts the connectedness of $S^n - g(E)$. □

Corollary 1.2. *If $f : \bar{D}^n \to D^n$ is a continuous injection, then $f(D^n)$ is open.*

Proof. Here, we will essentially just replicate the proof of Theorem 2B.3 in [H]. Fix some point $x \in S^n$. Since $S^n - x$ is open in S^n and homeomorphic to D^n, it is equivalent to consider a continuous injection $f : \bar{D}^n \to S^n$ with $x \notin f(\bar{D}^n)$ and show that $f(D^n)$ is open in S^n. Note that we have a disjoint union

$$S^n = f(D^n) \sqcup f(\partial D^n) \sqcup \big(S^n - f(\bar{D}^n)\big).$$

[1] By this, we mean a metric ball in S^{n-1} (of radius less than 2), using the standard metric inherited by viewing $S^{n-1} \subseteq \mathbb{R}^n$ as the unit sphere. These balls form a base for the metric topology on S^{n-1}, which is how we can guarantee the existence of V.

Since S^{n-1} is homeomorphic to ∂D^n, Theorem 1.1 implies that

$$S^n - f(\partial D^n) = f(D^n) \sqcup \big(S^n - f(\bar{D}^n)\big)$$

has exactly two connected components. As remarked in the proof of the theorem, each of these components is open in S^n. But Theorem 1.1 also implies that $S^n - f(\bar{D}^n)$ is connected. Since $f(D^n)$ is clearly also connected, these sets are the aforementioned components of $S^n - f(\partial D^n)$, so they are open in S^n. □

In the case where $n = 2$, the Jordan-Schoenflies theorem states not only that the complement $S^2 - g(S^1)$ has two components, but that the closure of each component is homeomorphic to $\bar{D}^2$. When $n > 2$, the analogous assertion may fail to hold, as demonstrated by pathologies like the Alexander horned sphere. But this generalization to higher dimensions does hold when g is "nice enough" in a sense that we will make precise later on (this is what we will ultimately prove in this note). As a first step, we can prove another result due to Brouwer, called *invariance of domain* (one form of which is the corollary we just proved). This is a theorem concerning manifolds, so we a recall a *weak* definition here:

Definition 1.3. *A topological space X is an n-manifold if every point $x \in X$ admits an open neighborhood $U \ni x$ that is homeomorphic to D^n.*[2]

If the reader is encountering manifolds for the first time in this definition, they may find the following exercise instructive: prove that any open subset of an n-manifold is an n-manifold (this will be used without comment later on).

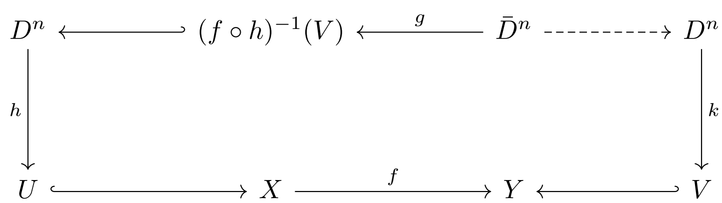
Proposition 1.4 (Invariance of Domain)**.** *Suppose X and Y are n-manifolds and $f : X \to Y$ is a continuous injection. Then $f(X) \subseteq Y$ is open.*

Proof. To prove that $f(X) \subseteq Y$ is open, we consider a point $x \in X$ and show that $f(X)$ contains a neighborhood of $f(x)$. Since X and Y are n-manifolds, there exist open sets $U \ni x$ and $V \ni f(x)$ and homeomorphisms $h : D^n \to U$ and $k : D^n \to V$. Let $\tilde{x} = h^{-1}(x)$ and notice that $(f \circ h)^{-1}(V) \subseteq D^n$ is an open set containing $\tilde{x}$. Therefore, there is a small embedded disk $g : \bar{D}^n \to D^n$ with

$$\tilde{x} \in g(D^n) \qquad \text{and} \qquad g(\bar{D}^n) \subseteq (f \circ h)^{-1}(V).$$

These maps are illustrated in the following diagram:

[2] It might be more appropriate to call such spaces *locally Euclidean* and reserve the title of manifold for those that are Hausdorff and second-countable, but the result of Proposition 1.4 does not actually require either of these assumptions.



Since $(f \circ h \circ g)(\bar{D}^n) \subseteq V$ and all of these maps are injective, we get a continuous injection $k^{-1} \circ f \circ h \circ g : \bar{D}^n \to D^n$. It follows from Corollary 1.2 that the set

$$(k^{-1} \circ f \circ h \circ g)(D^n) \subseteq D^n$$

is open and therefore that $(f \circ h \circ g)(D^n)$ is open in V. Since V is open in Y and $\tilde{x} \in g(D^n)$, it follows that $(f \circ h \circ g)(D^n) \subseteq f(X)$ is open in Y and contains the point $(f \circ h)(\tilde{x}) = f(x)$, as desired. □

The quintessential application of invariance of domain is in showing that the dimension of a manifold is well-defined, i.e. if a non-empty topological space X is an n-manifold and an m-manifold, then $n = m$. But we will only need to use it to directly identify open sets in various proofs.

2 Cellular Subsets

The key technical notion used in [B] is that of a cellular subset, for which it will be useful to establish two equivalent definitions:

Definition 2.1. *Let X be a metrizable n-manifold and consider a subset $C \subseteq X$. The following two conditions are equivalent:*

(a) *There is a sequence of embeddings $f_i : \bar{D}^n \to X$ with $f_{i+1}(\bar{D}^n) \subseteq f_i(D^n)$ for all $i \in \mathbb{N}$, which satisfy*

$$C = \bigcap_{i \in \mathbb{N}} f_i(\bar{D}^n).$$

(b) *For any open set $U \subseteq X$ with $C \subseteq U$, there is an injective map $g : \bar{D}^n \to U$, which satifies $C \subseteq g(D^n)$.*

If these equivalent conditions hold, we will say that C is cellular in X.

Proof. We need to prove that each condition implies the other.

(a) $\Longrightarrow$ (b) : Suppose the embeddings f_i satisfy (a) and let $U \subseteq X$ be an open set with $C \subseteq U$. For any $i \in \mathbb{N}$, we have $C \subseteq f_{i+1}(\bar{D}^n) \subseteq f_i(D^n)$.

Therefore, we just need to find some $i \in \mathbb{N}$ such that $f_i(\bar{D}^n) \subseteq U$. If no such $i \in \mathbb{N}$ existed, we would have a descending sequence

$$f_1(\bar{D}^n) - U \supseteq f_2(\bar{D}^n) - U \supseteq f_3(\bar{D}^n) - U \supseteq \dots$$

of non-empty, compact subsets. But then the intersection

$$\bigcap_{i \in \mathbb{N}} \big(f_i(\bar{D}^n) - U \big) = C - U$$

would be non-empty, contradicting the assumption that $C \subseteq U$.

(b) $\Longrightarrow$ (a) : If $C \neq \bar{C}$, we can choose some $x \in \bar{C} - C$ and set $U = X - x$. Then (b) yields an injective map $g : \bar{D}^n \to U$ with $C \subseteq g(D^n)$. Since $\bar{D}^n$ is compact and X is Hausdorff, the set $g(\bar{D}^n)$ is closed in X and thus $x \in \bar{C} \subseteq g(\bar{D}^n)$, which contradicts $g(\bar{D}^n) \subseteq U$. Thus $C = \bar{C}$ is closed in X.

Now let d be a metrization of X (i.e. a metric inducing the given topology on X). For each $i \in \mathbb{N}$, the set

$$V_i = \big\{ x \in X : d(x, C) < \tfrac{1}{i} \big\} = \bigcup_{c \in C} \big\{ x \in X : d(x, c) < \tfrac{1}{i} \big\}$$

is clearly open and contains C. These sets also clearly satisfy

$$\bigcap_{i \in \mathbb{N}} V_i = \bar{C} = C.$$

We inductively define maps $f_i : \bar{D}^n \to X$ and open sets $U_i \supseteq C$, as follows. Let $U_0 = X$. For any $i \geq 1$, we define $f_i : \bar{D}^n \to U_{i-1}$ to be the injective map given by (b) and let $U_i = f_i(D^n) \cap V_i$. For this method to work, each U_i must be open and contain C. This is obvious for U_0. Whenever $i \geq 1$, we can see that $f_i(D^n)$ is open by invariance of domain and contains C by the statement of (b) that was used to define f_i. But we already remarked that the same is true of each V_i, so it is true of the intersections U_i. Notice that $f_{i+1}(\bar{D}^n) \subseteq U_i \subseteq f_i(D^n)$ by definition and that

$$C \subseteq \bigcap_{i \in \mathbb{N}} f_{i+1}(\bar{D}^n) \subseteq \bigcap_{i \in \mathbb{N}} U_i \subseteq \bigcap_{i \in \mathbb{N}} V_i = C.$$

Because each $f_i : \bar{D}^n \to X$ is an injective map from a compact space to a Hausdorff space, it is necessarily an embedding. □

From the second definition of a cellular subset, we can see that this notion is hereditary with respect to the ambient manifold, in the following sense:

Corollary 2.2. *Let X be any metrizable n-manifold and let $V \subseteq X$ be open. Then V is a metrizable n-manifold and a subset $C \subseteq V$ is cellular in V if and only if it is cellular in X.*

We leave the proof of this corollary as an exercise for the reader and move on to demonstrate the existence of various self-homeomorphisms of $\bar{D}^n$ and S^n, which do not directly involve cellularity, but will be of use later. Let $\mathrm{Map}_\partial(\bar{D}^n)$ denote the set of maps $f : \bar{D}^n \to \bar{D}^n$ that fix ∂D^n pointwise and let

$$\mathrm{Homeo}_\partial(\bar{D}^n) = \big\{ f \in \mathrm{Map}_\partial(\bar{D}^n) : f \text{ is a homemorphism} \big\}.$$

Lemma 2.3. *For any $p \in D^n$, there exists $\psi \in \mathrm{Homeo}_\partial(\bar{D}^n)$ with $\psi(0) = p$.*[3]

Proof. We first define a map $\Psi : S^{n-1} \times [0,1] \to \bar{D}^n$ by $\Psi(v,t) = p + t(v-p)$. This satisfies $\Psi^{-1}(p) = S^{n-1} \times 0$. To prove that this is the only inverse set, suppose that $t_1, t_2 > 0$ and $\Psi(v_1,t_1) = \Psi(v_2,t_2)$. Then $t_1 v_1 - t_2 v_2 = (t_1 - t_2)p$, so if $t_1 = t_2$, then we may conclude that $v_1 = v_2$. For the sake of contradiction, suppose that $t_1 \neq t_2$. Then the reverse triangle inequality implies that

$$\|p\| = \frac{\|t_1 v_1 - t_2 v_2\|}{|t_1 - t_2|} \geq \frac{\big|t_1\|v_1\| - t_2\|v_2\|\big|}{|t_1 - t_2|} = \frac{|t_1 - t_2|}{|t_1 - t_2|} = 1.$$

But this contradicts our assumption that $p \in D^n$. Hence, the map Ψ is injective on $S^{n-1} \times (0,1]$. Now consider an arbitrary point $w \in \bar{D}^n - p$ and define

$$f(t) = \big\| \tfrac{1}{t}(w-p) + p \big\|$$

This defines a map $f : (0,1] \to [0,\infty)$ with $f(1) = \|w\| \leq 1$. We also have

$$t \leq \frac{\|w-p\|}{1 + \|p\|} \quad \Longrightarrow \quad \begin{aligned} f(t) &= \big\| \tfrac{1}{t}(w-p) + p \big\| \\ &\geq \tfrac{1}{t}\|w-p\| - \|p\| \\ &\geq 1 + \|p\| - \|p\| = 1. \end{aligned}$$

By the intermediate value theorem, there exists some $t \in (0,1]$ with $f(t) = 1$. We can then set $v = \frac{1}{t}(w-p) + p \in S^{n-1}$ and immediately see that $\Psi(v,t) = w$. Hence, the map Ψ is surjective. Define the quotient map $q : S^{n-1} \times [0,1] \to \bar{D}^n$ by $q(v,t) = tv$. Since q simply collapses $S^{n-1} \times 0$ to a point, the map Ψ descends along the quotient map q to define a bijective map $\psi : \bar{D}^n \to \bar{D}^n$ with $\psi(0) = p$. Since $\bar{D}^n$ is compact, this map ψ is a homeomorphism. Given any $v \in S^{n-1}$, we clearly have $q(v,1) = v = \Psi(v,1)$, so ψ fixes $S^{n-1} = \partial D^n$ pointwise. □

[3]Here, we are viewing $\bar{D}^n \subseteq \mathbb{R}^n$ as the closed unit ball under the usual norm $\|\cdot\|$ on $\mathbb{R}^n$ and $S^{n-1} = \partial D^n \subseteq \mathbb{R}^n$ as the unit sphere.

Lemma 2.4. *Suppose $U, Q \subseteq D^n$ are open and $\bar{Q} \cap \partial D^n = \emptyset$ (where $\bar{Q}$ refers to the closure in $\bar{D}^n$). Given any $p \in U \cap Q$, there exists some $\varphi \in \mathrm{Homeo}_\partial(\bar{D}^n)$ that restricts to the identity on a neighborhood of p and satisfies $\varphi(\bar{Q}) \subseteq U$.*

Proof. Applying Lemma 2.3, we may assume that $p = 0$. Since the usual norm $\nu : \mathbb{R}^n \to [0,1]$ is a continuous map,[4] the compact set

$$\bar{Q} \subseteq D^n = \nu^{-1}[0,1)$$

admits some $M \in (0,1)$ such that $\nu(\bar{Q}) \subseteq [0,M]$. Since $U \in D^n$ is an open set containing 0, there is some small radius $m \in (0,M)$ with $\nu^{-1}[0,m) \subseteq U$. Next, we can define an increasing, piecewise-linear homeomorphism $\mu : [0,1] \to [0,1]$ by linear interpolation between the points $(0,0)$, $\big(\frac{m}{2}, \frac{m}{2}\big)$, $\big(\frac{M+1}{2}, m\big)$ and $(1,1)$. Letting $a = \frac{m}{m-M-1}$ and $b = \frac{m-1}{M-1}$, we can make this explicit by writing

$$\mu(t) = \begin{cases} t, & 0 \leq 2t \leq m \\ -ax + \frac{m}{2}(a+1), & m \leq 2t \leq M+1 \\ 2bx + 1 - 2b, & M+1 \leq 2t \leq 2 \end{cases}$$

This is defined so that $\mu(t) = t$ whenever $t < \frac{m}{2}$ and $\mu(t) < m$ whenever $t \leq M$, because $M < \frac{M+1}{2}$ and $\mu\big(\frac{M+1}{2}\big) = m$. As in the previous proof, we also consider the quotient map $q : S^{n-1} \times [0,1] \to \bar{D}^n$ given by $q(v,t) = tv$, which collapses $S^{n-1} \times 0$ to a point. Now consider the following diagram:

$$\begin{array}{ccc} S^{n-1} \times [0,1] & \xrightarrow{\mathrm{Id}\times\mu} & S^{n-1} \times [0,1] \\ \Big\downarrow q & & \Big\downarrow q \\ \bar{D}^n & \overset{\varphi}{\dashrightarrow} & \bar{D}^n \end{array}$$

Because $\mathrm{Id} \times \mu$ is a homeomorphism that restricts to the identity on $S^{n-1} \times 0$, it descends along q to a homeomorphism $\varphi : \bar{D}^n \to \bar{D}^n$. Since $\mathrm{Id} \times \mu$ restricts to the identity on $S^{n-1} \times \big[0, \frac{m}{2}\big)$, we can see that φ restricts to the identity on

$$q\Big(S^{n-1} \times \big[0, \tfrac{m}{2}\big)\Big) = \nu^{-1}\big[0, \tfrac{m}{2}\big),$$

which is a neighborhood of 0. Next, recall that

$$\bar{Q} \subseteq \nu^{-1}[0,M] = q\big(S^{n-1} \times [0,M]\big) \quad \text{and} \quad q\big(S^{n-1} \times [0,m)\big) = \nu^{-1}[0,m) \subseteq U.$$

[4] In this proof, we write ν instead of $\|\cdot\|$, so that it is easier to notate inverse images.

Since $\mu\big([0,M]\big) \subseteq [0,m)$, we can conclude that

$$\begin{aligned}\varphi(\bar{Q}) &\subseteq \varphi \circ q\big(S^{n-1} \times [0,M]\big) \\ &= q \circ (\mathrm{Id} \times \mu)\big(S^{n-1} \times [0,M]\big) \\ &\subseteq q\big(S^{n-1} \times [0,m)\big) \subseteq U. \end{aligned}$$ ☐

Corollary 2.5. *Suppose $U, Q \subseteq S^n$ are open and $\bar{Q} \neq S^n$. For any $p \in U \cap Q$, there exists some homeomorphism $\varphi : S^n \to S^n$ that restricts to the identity on a neighborhood of p and satisfies $\varphi(\bar{Q}) \subseteq U$.*

Proof. Choose a point $x \in S^n - \bar{Q}$. Since S^n can be viewed as a quotient of $\bar{D}^n$ given by collapsing ∂D^n to a point, we can find a quotient map $\pi : \bar{D}^n \to S^n$ where $\partial D^n = \pi^{-1}(x)$ is the only inverse set. We can then define

$$p_0 = \pi^{-1}(p), \quad U_0 = \pi^{-1}(U) \cap D^n \quad \text{and} \quad Q_0 = \pi^{-1}(Q).$$

Because $\bar{Q}_0$ is compact and S^n is Hausdorff, we can see that $\pi(\bar{Q}_0) = \bar{Q}$.[5] Along with our initial assumption that $x \notin \bar{Q}$, this implies that $\bar{Q}_0 \cap \partial D^n = \emptyset$. It is also straightforward to check that $U_0, Q_0 \subseteq D^n$ are open and $p_0 \in U_0 \cap Q_0$, so Lemma 2.4 yields some $\varphi_0 \in \mathrm{Homeo}_\partial(\bar{D}^n)$ that restricts to the identity on a neighborhood of p_0 and satisfies $\varphi_0(\bar{Q}_0) \subseteq U_0$. Consider the following diagram:

$$\begin{array}{ccc} \bar{D}^n & \xrightarrow{\ \varphi_0\ } & \bar{D}^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S^n & \overset{\varphi}{\dashrightarrow} & S^n \end{array}$$

Since the map φ_0 restricts to the identity on ∂D^n, it descends along π to define a homeomorphism $\varphi : S^n \to S^n$ that restricts to the identity on a neighborhood of $\pi(p_0) = p$.[6] Finally, using the fact that $\bar{Q} = \pi(\bar{Q}_0)$, we have

$$\varphi(\bar{Q}) = (\varphi \circ \pi)(\bar{Q}_0) = (\pi \circ \varphi_0)(\bar{Q}_0) \subseteq \pi(U_0) \subseteq U.$$ ☐

3 Bing Shrinking

Cellularity plays a key role, because the topology of a manifold is unchanged when a cellular subset is collapsed to a single point (this phenomenon is called "Bing shrinking"). We will stop just short of proving this fact in full generality.

[5] Since π is surjective, we have $\pi(Q_0) = \pi\big(\pi^{-1}(Q)\big) = Q$ and thus $\pi(\bar{Q}_0) \subseteq \bar{Q}$, by continuity. Conversely, we have $Q = \pi(Q_0) \subseteq \pi(\bar{Q}_0)$ and thus $\bar{Q} \subseteq \pi(\bar{Q}_0)$. This shows that $\pi(\bar{Q}_0) = \bar{Q}$.

[6] If $V_0 \subseteq D^n$ is a neighborhood of p_0 on which φ_0 restricts to the identity, then φ restricts to the identity on $\pi(V_0) \subseteq S^n$, which is open because V_0 is saturated with respect to π.

Proposition 3.1. *If $C \subseteq D^n$ is a cellular subset, then there exists a surjection $f \in \mathrm{Map}_\partial(\bar{D}^n)$ such that C is the only inverse set under f.*[7]

Proof. Pick a point $p \in C$. Since C is cellular in D^n, there exists a sequence of embeddings $f_i : \bar{D}^n \to D^n$ with $f_{i+1}(\bar{D}^n) \subseteq f_i(D^n)$ for all $i \in \mathbb{N}$, which satisfy

$$C = \bigcap_{i \in \mathbb{N}} f_i(\bar{D}^n).$$

We also define open sets $V_i = \{x \in D^n : \|x - p\| < 2^{1-i}\}$, where $\|\cdot\|$ is the usual norm on $\bar{D}^n$. We now inductively define homeomorphisms $h_i \in \mathrm{Homeo}_\partial(\bar{D}^n)$, as follows. Let $h_0 = \mathrm{Id}$. Given $i \geq 1$, we will define h_i in terms of h_{i-1} so that:

- $h_i(p) = p$;
- $h_i \circ f_{i+1}(D^n) \subseteq V_i$;
- $h_i = h_{i-1}$ on $\bar{D}^n - f_i(D^n)$.

Note that h_0 indeed satisfies the first two conditions, since $f_1(D^n) \subseteq D^n = V_0$. Since $p \in C \subseteq f_{i+1}(\bar{D}^n) \subseteq f_i(D^n)$, we may consider the point $p_i = f_i^{-1}(p) \in D^n$ and define an open neighborhood of p_i by

$$U_i = (h_{i-1} \circ f_i)^{-1}(V_i) \cap D^n \subseteq D^n$$

By invariance of domain, we know that $f_{i+1}(D^n)$ is open in D^n, so we use the fact that $p \in f_{i+1}(D^n) \subseteq f_i(D^n)$ to define another open neighborhood of p_i by

$$Q_i = f_i^{-1}\big(f_{i+1}(D^n)\big) \subseteq D^n$$

Since $\bar{D}^n$ is compact, we can also see that $f_{i+1}(\bar{D}^n) \subseteq f_i(D^n)$ is closed in $\bar{D}^n$ and thus that $f_i^{-1}\big(f_{i+1}(\bar{D}^n)\big) \subseteq D^n$ is closed. Then $\bar{Q}_i \cap \partial D^n = \emptyset$, because

$$\bar{Q}_i \subseteq f_i^{-1}\big(f_{i+1}(\bar{D}^n)\big) \subseteq D^n$$

Since $p_i \in U_i \cap Q_i$, Lemma 2.4 yields a homeomorphism $g_i \in \mathrm{Homeo}_\partial(\bar{D}^n)$ that satisfies $g_i(p_i) = p_i$ and $g_i(\bar{Q}_i) \subseteq U_i$. It follows that

$$f_i \circ g_i \circ f_i^{-1} : f_i(\bar{D}^n) \to f_i(\bar{D}^n)$$

[7] It turns out that every $f \in \mathrm{Map}_\partial(\bar{D}^n)$ is surjective, but proving this requires homology, so we avoid it here. The gist of the proof is as follows: If $p \notin f(\bar{D}^n)$, we can describe a retract

$$g : \bar{D}^n - p \to \partial D^n.$$

Then we get a retract $g \circ f : \bar{D}^n \to \partial D^n$, but no such map exists (see Corollary 2.15 in [H]).

is a homeomorphism that fixes $f_i(\partial D^n) \cup \{p\}$ pointwise. We also consider

$$\mathrm{Id} : \bar{D}^n - f_i(D^n) \to \bar{D}^n - f_i(D^n)$$

Note that the two domains $f_i(\bar{D}^n)$ and $\bar{D}^n - f_i(D^n)$ form a closed cover of $\bar{D}^n$ and that these two homeomorphisms agree on $f_i(\partial D^n)$, which is the intersection of their domains, so they glue together to give a homeomorphism $\tilde{g}_i : \bar{D}^n \to \bar{D}^n$. Since $f_i(\bar{D}^n) \subseteq D^n$, we know that $\tilde{g}_i$ fixes ∂D^n pointwise, so $\tilde{g}_i \in \mathrm{Homeo}_\partial(\bar{D}^n)$. We now define $h_i = h_{i-1} \circ \tilde{g}_i \in \mathrm{Homeo}_\partial(\bar{D}^n)$. Since $\tilde{g}_i$ fixes p by construction and h_{i-1} fixes p by the inductive hypothesis, we have $h_i(p) = p$. The fact that $h_i = h_{i-1}$ on $\bar{D}^n - f_i(D^n)$ is immediate, because $\tilde{g}_i = \mathrm{Id}$ on this same domain. Lastly, since $f_{i+1}(\bar{D}^n) \subseteq f_i(D^n)$, we have

$$\begin{aligned} h_i \circ f_{i+1}(D^n) &= h_{i-1} \circ \tilde{g}_i \circ f_{i+1}(D^n) \\ &= h_{i-1} \circ f_i \circ g_i \circ f_i^{-1} \circ f_{i+1}(D^n) \\ &= h_{i-1} \circ f_i \circ g_i(Q_i) \\ &\subseteq h_{i-1} \circ f_i(U_i) \subseteq V_i \end{aligned}$$

We have now verified the desired properties of h_i, completing the inductive step.

Since h_{i-1} and h_i are bijections $\bar{D}^n \to \bar{D}^n$, these properties imply that

$$\begin{aligned} h_i \circ f_i(D^n) &= \bar{D}^n - h_i\big(\bar{D}^n - f_i(D^n)\big) \\ &= \bar{D}^n - h_{i-1}\big(\bar{D}^n - f_i(D^n)\big) \\ &= h_{i-1} \circ f_i(D^n) \subseteq V_{i-1} \end{aligned}$$

Thus $\|h_i - h_{i-1}\| \leq 2^{3-i}$ on $f_i(D^n)$, because for any $x \in f_i(D^n)$, we have

$$\|h_i(x) - h_{i-1}(x)\| \leq \|h_i(x) - p\| + \|h_{i-1}(x) - p\| < 2^{2-i} + 2^{2-i} = 2^{3-i}$$

But we also have $h_i = h_{i-1}$ on $\bar{D}^n - f_i(D^n)$, so the inequality $\|h_i - h_{i-1}\| \leq 2^{3-i}$ holds on all of $\bar{D}^n$. For any $k \in \mathbb{N}$, we then have

$$\begin{aligned} \|h_{i+k} - h_i\| = \left\| \sum_{j=1}^{k} (h_{i+j} - h_{i+j-1}) \right\| &\leq \sum_{j=1}^{k} \|h_{i+j} - h_{i+j-1}\| \\ &\leq \sum_{j=1}^{k} 2^{3-i-j} = 2^{3-i}(1 - 2^{-k}) < 2^{3-i} \end{aligned}$$

This shows that the sequence of maps $h_i \in \mathrm{Homeo}_\partial(\bar{D}^n)$ is uniformly Cauchy, so it converges to a map $f \in \mathrm{Map}_\partial(\bar{D}^n)$.[8] It remains to show that f is surjective and that C is its only inverse set.

[8] Since each $h_i \in \mathrm{Homeo}_\partial(\bar{D}^n)$ fixes ∂D^n pointwise, the same is true of their limit f.

If $x \in \bar{D}^n - f_{i+1}(D^n)$, then $x \in \bar{D}^n - f_j(D^n)$ for any $j \geq i+1$, so we get

$$h_i(x) = h_{i+1}(x) = \cdots = h_j(x).$$

It follows that $f = h_i$ on $\bar{D}^n - f_{i+1}(D^n)$. But if $y \in C$, then we get $y \in f_{j+1}(D^n)$ and thus $h_j(y) \in V_j$ for all $j \in \mathbb{N}$. This means that $\|h_j(y) - p\| < 2^{1-j}$ and thus

$$f(y) = \lim_{j \to \infty} h_j(y) = p.$$

Hence $f(C) = \{p\}$. Given distinct points $x, y \in \bar{D}^n - C$, we have $x \notin f_{i+1}(D^n)$ and $y \notin f_{j+1}(D^n)$ for some $i, j \in \mathbb{N}$. If $k = \max\{i, j\}$, we have $x, y \notin f_{k+1}(D^n)$ and thus

$$f(x) = h_k(x) \neq h_k(y) = f(y) \qquad \text{and} \qquad f(x) = h_k(x) \neq h_k(p) = p,$$

because h_k is injective and $x \neq p$. Thus $f^{-1}(p) = C$ and the restriction

$$f : \bar{D}^n - C \to \bar{D}^n - p$$

is injective, so C is the only inverse set. Given any $z \in \bar{D}^n - p$, we have $z \notin V_i$ for some $i \in \mathbb{N}$, which implies that $h_i^{-1}(z) \notin f_{i+1}(D^n)$ and thus $f \circ h_i^{-1}(z) = z$. This shows that f is surjective, finally completing the proof. □

Since $\bar{D}^n$ is compact and Hausdorff, any surjection $f \in \mathrm{Map}_\partial(\bar{D}^n)$ is closed and thus a quotient map, so this proposition implies that collapsing C to a point results in a quotient space homeomorphic to the original space $\bar{D}^n$. We can also turn this around to prove that a certain space having $\bar{D}^n$ as a quotient (under a quotient map with certain specific properties) is itself homeomorphic to $\bar{D}^n$:

Lemma 3.2. *Suppose that $Q \subseteq S^n$ is open and $f : \bar{Q} \to \bar{D}^n$ is a surjective map with exactly one inverse set C, such that $f(\partial Q) = \partial D^n$. If C is cellular in Q, then there exists a homeomorphism $h : \bar{Q} \to \bar{D}^n$ with $h(\partial Q) = \partial D^n$.*

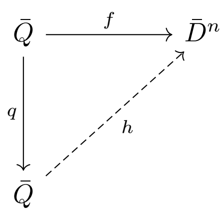
Proof. Since C is a cellular subset of Q, there exists an embedding $k : \bar{D}^n \to Q$ such that $C \subseteq k(D^n)$. Note that $k(D^n) \subseteq Q$ is open by invariance of domain, so C is cellular in $k(D^n)$ by Corollary 2.2 and hence $k^{-1}(C)$ is cellular in D^n. By Proposition 3.1, there exists a surjection $p \in \mathrm{Map}_\partial(\bar{D}^n)$ such that $k^{-1}(C)$ is the only inverse set under p. It follows that

$$k \circ p \circ k^{-1} : k(\bar{D}^n) \to k(\bar{D}^n)$$

is a surjective map that fixes $k(\partial D^n)$ pointwise, where C is the only inverse set. We also consider the following identity map:

$$\mathrm{Id} : \bar{Q} - k(D^n) \to \bar{Q} - k(D^n)$$

Notice that the two domains $k(\bar{D}^n)$ and $\bar{Q} - k(D^n)$ form a closed cover of $\bar{Q}$ and these two maps agree on $k(\partial D^n)$, which is the intersection of the domains. Therefore, they glue together to define a map $q : \bar{Q} \to \bar{Q}$. It is straightforward to confirm that q is surjective and that C is the only inverse set. Notice that $\bar{Q}$ is compact, being a closed subset of S^n, so q is closed and thus a quotient map. Now consider the following diagram:



Since C is the only inverse set under either f or q, the surjective map f descends to a bijective function $h : \bar{Q} \to \bar{D}^n$ satisfying $f = h \circ q$. But q is a quotient map, so h is continuous. Because $\bar{Q}$ is compact, the continuous bijection $h : \bar{Q} \to \bar{D}^n$ is a homeomorphism. Since Q is open in S^n, we have

$$\partial Q = \bar{Q} - Q \subseteq \bar{Q} - k(D^n).$$

Thus q restricts to the identity on ∂Q, so $h(\partial Q) = f(\partial Q) = \partial D^n$. □

We conclude this section with one more technical lemma regarding quotient maps like those in Proposition 3.1, which will be needed in our ultimate proof:

Lemma 3.3. *If Z is a T_4 space (normal and Hausdorff) and $A \subseteq Z$ is closed, then the quotient space Z/A (given by collapsing A to a point) is also T_4.*

Proof. Let $p : Z \to Z/A$ denote the quotient map. Every point in Z/A is closed, because every fiber of the quotient map p is closed. Notice that a set $B \subseteq Z$ is saturated with respect to p if and only if $A \subseteq B$ or $A \cap B = \emptyset$. Consider disjoint closed sets $K_1, K_2 \subseteq Z/A$. Then $p^{-1}(K_1), p^{-1}(K_2) \subseteq Z$ are closed, disjoint and saturated. Since Z is normal, there exist disjoint open sets $U_1, U_2 \subseteq Z$ separating $p^{-1}(K_1)$ and $p^{-1}(K_2)$. For $i = 1$ or 2, we now define

$$V_i = \begin{cases} U_i - A, & A \cap p^{-1}(K_i) = \emptyset \\ U_i, & A \subseteq p^{-1}(K_i) \end{cases}$$

Then V_i is an open neighborhood of $p^{-1}(K_i)$, satisfying $A \subseteq V_i$ or $A \cap V_i = \emptyset$. We also have $V_1 \cap V_2 \subseteq U_1 \cap U_2 = \emptyset$, so V_1 and V_2 are saturated, disjoint open sets in Z separating $p^{-1}(K_1)$ and $p^{-1}(K_2)$. It follows that $p(V_1)$ and $p(V_2)$ are disjoint open sets in Z/A separating K_1 and K_2, as desired. □

4 Detecting Cellularity

To apply Lemma 3.2, we will need to be able to identify certain cellular subsets of the sphere. First, we prove a small lemma on the connectivity of manifolds:

Lemma 4.1. *Let X be a connected n-manifold with $n > 1$. If $P \subseteq X$ is finite, then the complement $X - P$ is also connected.*

Proof. Since points in X are closed,[9] deleting any finite set of points results in an open subset of X, which is also an n-manifold. As such, we may proceed by induction and assume that $P = \{p\}$ is a singleton. If $X = D^n$, then Lemma 2.3 yields a homeomorphism $\psi : D^n \to X$ with $\psi(0) = p$. Since $D^n - 0$ is connected (being homeomorphic to $S^{n-1} \times (0, 1]$), it follows that $X - p$ is connected.

In general, suppose for the sake of contradiction that $X - p$ is disconnected. Then we can write $X - p = V \sqcup W$, where the sets V and W are open in $X - p$ and thus in X. Since X is an n-manifold, there is an open neighborhood $U \ni p$ homeomorphic to D^n. Then $U - p$ is connected, by the case considered above, so it must lie entirely in either V or W. But if $U - p \subseteq V$, then $U \cup V = \{p\} \sqcup V$ is open in X, so the connectedness of X is contradicted by the decomposition

$$X = \big(\{p\} \sqcup V\big) \sqcup W.$$

If $U - p \subseteq W$, we get an analogous contradiction. Thus $X - p$ is connected. □

With this lemma in hand, we can now proceed to the meat of this section, which is an amalgam of Theorem 0 in [B] and Lemma 4.2 in [P].

Lemma 4.2. *Suppose $f : \bar{D}^n \to S^n$ is a map with finitely many inverse sets, all of which lie in D^n. Denote these inverse sets by $C_1, \dots, C_k$ and let $c_i \in S^n$ denote the point with $C_i = f^{-1}(c_i)$. Then we have:*

(a) $f(D^n)$ is a connected component of $S^n - f(\partial D^n)$, which we denote by E.

(b) Let $I = C_1 \cup \cdots \cup C_k = \{p \in \bar{D}^n : f^{-1}\big(f(p)\big)$ contains multiple points$\}$. Then f restricts to a quotient map $f : D^n \to E$ and to a homeomorphism

$$f : D^n - I \longrightarrow E - \{c_1, \dots, c_k\}.$$

(c) If $U \subseteq D^n$ is an open set that contains C_k and is disjoint from $I - C_k$, then there exists a map $g : \bar{D}^n \to U$ that restricts to the identity on C_k and whose inverse sets are precisely $C_1, \dots, C_{k-1}$.

[9] Even with our weak definition of manifolds, points are always closed. A proof of this fact is left as an exercise for the reader, since this lemma will only be used in situations where X is clearly Hausdorff.

(d) *The sets* $C_1, \dots, C_k$ *are each cellular in* D^n.

Proof. (a) Since all of the inverse sets of f lie in D^n, we can see that $f|_{\partial D^n}$ is injective and $f(D^n) \cap f(\partial D^n) = \emptyset$. Since ∂D^n is homeomorphic to S^{n-1}, the complement $S^n - f(\partial D^n)$ has two connected components, each with boundary $f(\partial D^n)$. Because $f(D^n)$ is connected and disjoint from $f(\partial D^n)$, it must lie in one of these connected components, which we call E. To prove the assertion that $f(D^n) = E$, it remains to show that $E - f(D^n) = \emptyset$.

Note that $I = C_1 \cup \dots \cup C_k$ is closed, so $D^n - I$ is open in D^n and thus is an n-manifold. Since f is injective on $D^n - I$, invariance of domain implies that $f(D^n - I)$ is open in S^n. Since $E \cap f(\partial D^n) = \emptyset$, we have

$$E - f(\bar{D}^n) = E - f(D^n).$$

Thus $E - f(D^n)$ is open in E (because $f(\bar{D}^n)$ is closed in S^n). Therefore

$$E - f(I) = \big(E - f(D^n)\big) \sqcup f(D^n - I)$$

is a partition of $E - f(I)$ into two open sets. But $f(I)$ is finite and $E \subseteq S^n$ is open and connected, so $E - f(I)$ is connected by Lemma 4.1. Therefore, either $E - f(D^n) = \emptyset$ or $f(D^n - I) = \emptyset$. But if we have $f(D^n - I) = \emptyset$, then $f(D^n) = f(I)$ is finite and therefore closed in S^n, which means that

$$f(\bar{D}^n) = f(D^n) \sqcup f(\partial D^n)$$

is a partition of $f(\bar{D}^n)$ into nonempty closed sets. But this contradicts the connectedness of $\bar{D}^n$. Instead, we must have $E - f(D^n) = \emptyset$, as desired.

(b) Since $\bar{D}^n$ is compact and S^n is Hausdorff, we can see that $f : \bar{D}^n \to f(\bar{D}^n)$ is closed and thus is a quotient map. Note that D^n and $D^n - I$ are both open and saturated (the former contains every inverse set, while the latter is disjoint from every inverse set), so the restrictions

$$f : D^n \longrightarrow f(D^n) \qquad \text{and} \qquad f : D^n - I \longrightarrow f(D^n - I)$$

are both quotient maps.[10] We know that $f(D^n) = E$ and it follows that

$$f(D^n - I) = E - \{c_1, \dots, c_k\}.$$

Since $D^n - I$ is disjoint from every inverse set of f, the restriction

$$f : D^n - I \to E - \{c_1, \dots, c_k\}$$

is an injective quotient map and therefore a homeomorphism.

---

[10] If $q : X \to Y$ is any quotient map and $U \subseteq X$ is open and saturated, then the restriction $q : U \to q(U)$ is again a quotient map (see Theorem 22.1 in [M]).

(c) Since U contains C_k and is disjoint from every other inverse set, we can see that U is saturated with respect to f. By part (b), it follows that $f(U)$ is open in E and thus in S^n. We also have $\bar{E} \neq S^n$ and

$$c_k \in f(U) \subseteq E - \{c_1, \ldots, c_{k-1}\} \subseteq E,$$

so Corollary 2.5 yields a homeomorphism $\varphi : S^n \to S^n$ with $\varphi(\bar{E}) \subseteq f(U)$, which restricts to the identity on an open set $W \subseteq S^n$ that contains c_k. Since $f(\bar{D}^n) \subseteq \bar{E}$ (by part (a) and the continuity of f) and $\varphi(c_k) = c_k$, we can see that

$$\varphi \circ f(\bar{D}^n - C_k) \subseteq \varphi(\bar{E} - \{c_k\}) \subseteq f(U) - \{c_k\} \subseteq E - \{c_1, \ldots, c_k\}.$$

By part (b), we then get a well-defined, injective map

$$f^{-1} \circ \varphi \circ f : \bar{D}^n - C_k \to D^n - I.$$

We also consider the following identity map:

$$\mathrm{Id} : f^{-1}(W) \to f^{-1}(W)$$

Because $c_k \in W$ and φ restricts to the identity on W, the two domains $\bar{D}^n - C_k$ and $f^{-1}(W)$ form an open cover of $\bar{D}^n$ and these two maps agree on the intersection of their domains. Therefore, they glue together to define a map $g : \bar{D}^n \to S^n$. Because $C_k \subseteq f^{-1}(W)$, the map g clearly restricts to the identity on C_k, so $g(C_k) = C_k \subseteq U$. Since U is saturated with respect to f, we also have

$$g(\bar{D}^n - C_k) = f^{-1} \circ \varphi \circ f(\bar{D}^n - C_k) \subseteq f^{-1}\big(f(U)\big) = U.$$

Thus $g(\bar{D}^n) \subseteq U$. It remains to determine the inverse sets of g. Note that

$$g(\bar{D}^n - C_k) \subseteq D^n - I \subseteq D^n - C_k.$$

Thus if $x \in C_k$, then $g^{-1}(x) \subseteq C_k$ and hence $g^{-1}\big(g(x)\big) = g^{-1}(x) = \{x\}$. Therefore, C_k does not intersect any inverse sets of g. In the composition

$$f^{-1} \circ \varphi \circ f : \bar{D}^n - C_k \to D^n - I,$$

the maps f^{-1} and φ are both homeomorphisms, so the inverse sets of g are just the intersections of $\bar{D}^n - C_k$ with the inverse sets of f. Therefore, the inverse sets of g are precisely $C_1, \ldots, C_{k-1}$.

(d) The claim is vacuous if $k = 0$, so we first assume that $k = 1$. Then $I = C_1$, so for any open set $U \subseteq D^n$ containing C_1, part (c) shows that there exists a map $g : \bar{D}^n \to U$ that restricts to the identity on C_1 and has no inverse sets. Thus g is injective and $C_1 = g(C_1) \subseteq g(D^n)$, so C_1 is cellular in D^n.

For $k > 1$, we proceed by induction. Letting $U = D^n - (C_1 \cup \dots C_{k-1})$, we may apply part (c) to get a map $g : \bar{D}^n \to U$ whose inverse sets are precisely $C_1, \dots, C_{k-1}$. Fix some embedding $h : D^n \to S^n$. Then the map $h \circ g : \bar{D}^n \to S^n$ has inverse sets $C_1, \dots, C_{k-1} \subseteq D^n$. By induction on k, we see that $C_1, \dots, C_{k-1}$ are each cellular in D^n. Swapping C_1 and C_k, the same argument shows that C_k is cellular in D^n. □

To conclude this section, we provide a slight rephrasing of a specific case of Lemma 4.2(d) that will be most relevant in our usage:

Lemma 4.3. *If $g : S^n \to S^n$ is a map with exactly two inverse sets B and C, then B and C are both cellular in S^n.*

Proof. Since B and C are both closed, the complement $S^n - (B \cup C)$ is open and non-empty (having $S^n = B \sqcup C$ would contradict the connectedness of S^n). Hence, we can find a standard (see footnote 1) open disk $V \subseteq S^n$ such that

$$\bar{V} \subseteq S^n - (B \cup C).$$

The complementary region $E = S^n - \bar{V}$ is an open set containing B and C, which admits a homeomorphism $h : \bar{D}^n \to \bar{E}$ with $h(D^n) = E$. Then the map

$$g \circ h : \bar{D}^n \to S^n$$

has precisely two inverse sets, $h^{-1}(B)$ and $h^{-1}(C)$, both of which lie in D^n. Hence, Lemma 4.2(d) implies that $h^{-1}(B)$ and $h^{-1}(C)$ are both cellular in D^n, so B and C are both cellular in E and therefore in S^n. □

5 Detecting Disks

We can now combine the tools that we have accumulated into a proof of the generalized Schoenflies theorem. Consider an embedding $g : S^{n-1} \times [0,1] \to S^n$. For brevity, given any subset $R \subseteq [0,1]$, we will write

$$S^{n-1}_R = g(S^{n-1} \times R).$$

Notice that if R is connected and $t \notin R$, then S^{n-1}_R is connected and therefore is contained in whichever component of $S^n - S^{n-1}_t$ it intersects. The main result that we will prove is:

Theorem 5.1 (Generalized Schoenflies Theorem)**.** *Let Q denote the connected component of $S^n - S_1^{n-1}$ that contains the connected set S_0^{n-1}. Then there exists a homeomorpism $h : \bar{Q} \to \bar{D}^n$ with $h(\partial Q) = \partial D^n$.*

Proof. In the course of the proof, we will need to name a couple other key subsets of the sphere. Let P denote the other connected component of $S^n - S_1^{n-1}$ (the one not containing S_0^{n-1}) and let C denote the connected component of $S^n - S_{1/2}^{n-1}$ not containing S_1^{n-1}. We next prove one last lemma:

Lemma 5.2. *The sets that we have defined satisfy the following relationships:*[11]

(a) $S_0^{n-1} \subseteq C$

Proof. If not, then the component of $S^n - S_{1/2}^{n-1}$ containing S_1^{n-1} must also contain S_0^{n-1} and therefore must contain $S_{[0,\frac{1}{2})\cup(\frac{1}{2},1]}^{n-1}$. It follows that $C \subseteq S^n - S_{1/2}^{n-1}$ is disjoint from $S_{(0,1)}^{n-1}$, which is a neighborhood of $S_{1/2}^{n-1}$ by invariance of domain. But this contradicts the fact that $S_{1/2}^{n-1} = \partial C$. □

(b) $\bar{C} \subseteq Q$

Proof. The closure $\bar{C} = C \cup S_{1/2}^{n-1}$ is connected and disjoint from S_1^{n-1}, so $\bar{C}$ lies in one of the connected components of $S^n - S_1^{n-1}$. But we know that S_0^{n-1} is a subset of both Q and $\bar{C}$, so we must have $\bar{C} \subseteq Q$. □

(c) $\bar{C} \cap S_{[0,1]}^{n-1} = S_{[0,\frac{1}{2}]}^{n-1}$

Proof. The containment $S_{[0,\frac{1}{2})}^{n-1} \subseteq C$ follows from $S_0^{n-1} \subseteq C$. The fact that $S_{[\frac{1}{2},1]}^{n-1}$ and C are disjoint follows from the definition of C as the component of $S^n - S_{1/2}^{n-1}$ *not* containing S_1^{n-1}. This implies that $C \cap S_{[0,1]}^{n-1} = S_{[0,\frac{1}{2})}^{n-1}$ and therefore that

$$\bar{C} \cap S_{[0,1]}^{n-1} = \left(C \cup S_{1/2}^{n-1}\right) \cap S_{[0,1]}^{n-1} = S_{[0,\frac{1}{2})}^{n-1} \cup S_{1/2}^{n-1} = S_{[0,\frac{1}{2}]}^{n-1}$$ □

(d) $\bar{P} \cap S_{[0,1]}^{n-1} = S_1^{n-1}$

Proof. Since P and Q are the components of $S^n - S_1^{n-1}$, we have

$$S^n = P \sqcup Q \sqcup S_1^{n-1}.$$

By assumption, we have $S_0^{n-1} \subseteq Q$ and thus $S_{[0,1)}^{n-1} \subseteq Q$. This implies that P and $S_{[0,1]}^{n-1}$ are disjoint, so we get

$$\bar{P} \cap S_{[0,1]}^{n-1} = (P \cup S_1^{n-1}) \cap S_{[0,1]}^{n-1} = S_1^{n-1}$$ □

[11] To not get bogged down in the details, it may help to skip the proofs of these assertions, at least on a first pass.

(e) $Q = \bar{C} \cup S^{n-1}_{[0,1)}$

Proof. Since Q is connected and it contains the set $\bar{C} \cup S^{n-1}_{[0,1)}$ by part (b), we need only show that this set is clopen in Q. The set

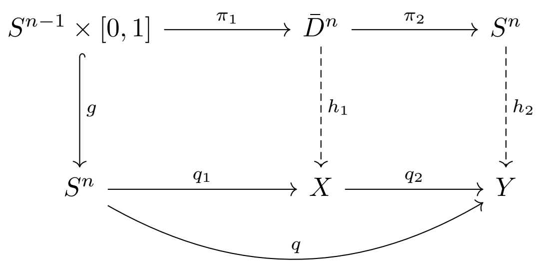
$$Q \cap \left(\bar{C} \cup S^{n-1}_{[0,1]}\right) = \bar{C} \cup S^{n-1}_{[0,1)}$$

is certainly closed in Q. Since $S^{n-1}_0 \subseteq C$ and $\bar{C} = C \cup S^{n-1}_{1/2}$, we also have

$$C \cup S^{n-1}_{(0,1)} = \bar{C} \cup S^{n-1}_{[0,1)}.$$

This set is open in S^n and thus in Q, since $S^{n-1}_{(0,1)}$ and C are open in S^n. $\square$

We now define several maps that are organized into a commutative diagram:



The maps π_1, π_2, q_1 and q_2 are each quotient maps given by collapsing a closed set to a point: π_1 collapses $S^{n-1} \times [0, \frac{1}{2}]$ and π_2 collapses $\partial D^n = \pi_1(S^{n-1} \times 1)$; q_1 collapses $\bar{C}$ and q_2 further collapses $q_1(\bar{P})$. The maps h_1 and h_2 are injections descending from g, which exist by parts (c) and (d) of Lemma 5.2. For brevity, we also write $q = q_2 \circ q_1$. Note also that the quotient spaces X and Y are both Hausdorff by Lemma 3.3.

Notice that Lemma 5.2(e) implies that $\bar{Q} = Q \cup S^{n-1}_1 = \bar{C} \cup S^{n-1}_{[0,1]}$. Since q_1 collapses $\bar{C}$ to a point and $\bar{C} \cap S^{n-1}_{[0,1]} \neq \emptyset$, we have $q_1(\bar{C}) \subseteq q_1\left(S^{n-1}_{[0,1]}\right)$ and thus

$$\begin{aligned} q_1(\bar{Q}) &= q_1(\bar{C}) \cup q_1\left(S^{n-1}_{[0,1]}\right) \\ &= q_1\left(S^{n-1}_{[0,1]}\right) = q_1 \circ g\left(S^{n-1} \times [0,1]\right) \\ &= h_1 \circ \pi_1\left(S^{n-1} \times [0,1]\right) = h_1(\bar{D}^n). \end{aligned}$$

Since q collapses $\bar{P}$ to a point and $\bar{P} \cap \bar{Q} \neq \emptyset$, we similarly have $q(\bar{P}) \subseteq q(\bar{Q})$. Combining this with $\bar{P} \cup \bar{Q} = S^n$ and the previous observation, we can see that

$$\begin{aligned} Y = q(S^n) &= q(\bar{P}) \cup q(\bar{Q}) = q(\bar{Q}) \\ &= q_2 \circ q_1(\bar{Q}) = q_2 \circ h_1(\bar{D}^n) \\ &= h_2 \circ \pi_2(\bar{D}^n) = h_2(S^n). \end{aligned}$$

Thus h_2 is a bijection, while h_1 is an injection with image $q_1(\bar{Q})$.

Since $\bar{D}^n$ and S^n are compact, while X and Y are Hausdorff, it follows that h_2 is a homeomorphism and h_1 is an embedding onto $q_1(\bar{Q})$. Hence, the map

$$h_2^{-1} \circ q : S^n \to S^n$$

is well-defined and has exactly two inverse sets: $\bar{C}$ and $\bar{P}$.[12] Lemmas 4.3 and 2.2 then imply that $\bar{C}$ is cellular in S^n and thus in Q. We also consider the map

$$f = h_1^{-1} \circ q_1 : \bar{Q} \to \bar{D}^n$$

Since $h_1 : \bar{D}^n \to q_1(\bar{Q})$ is a homeomorphism, this map is well-defined, surjective, and has exactly one inverse set $\bar{C}$ (because $\bar{C} \subseteq Q$). Notice also that

$$\begin{aligned} f(\partial Q) &= f(S_1^{n-1}) = f \circ g(S^{n-1} \times 1) \\ &= h_1^{-1} \circ q_1 \circ g(S^{n-1} \times 1) \\ &= \pi_1(S^{n-1} \times 1) = \partial D^n \end{aligned}$$

Thus, the desired homeomorphism $h : \bar{Q} \to \bar{D}^n$ is guaranteed by Lemma 3.2. $\square$

References

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[12] This uses the fact that $\bar{C} \cap \bar{P} = \emptyset$, which is an immediate consequence of Lemma 5.2(b).