# The Local Version of Ehresmann's Theorem

Consider a submersion $f : M \to N$. The vertical distribution of $f$ is the sub-bundle

$$V = \ker(Df) \subseteq TM.$$

A horizontal distribution is any sub-bundle $H \subseteq TM$ complementary to $V$, which means that $V \oplus H = TM$ (such a sub-bundle always exists; for example, if we put a Riemannian metric on $M$, we can set $H = V^{\perp}$). The restricted tangent mapping depicted below is a fiberwise isomorphism:

$$\begin{array}{ccc} H & \xrightarrow{\;Df\;} & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\;f\;} & N \end{array}$$

Thus any vector field $X \in \mathfrak{X}(N)$ has a unique horizontal lift, i.e. a vector field $\widetilde{X} \in \mathfrak{X}(M)$ that is everywhere tangent to $H$ and $f$-related to $X$. We will use this lifting property to prove the following theorem:

**Theorem.** For any $x \in M$, let $F_x$ denote the connected component of $f^{-1}\big(f(x)\big)$. For any $x \in M$ such that $F_x$ is compact, there exists a open set $U \ni f(x)$ and an open embedding $\Phi : U \times F_x \to M$, which satisfies:

- $f \circ \Phi(c, y) = c$ for all $c \in U$ and $y \in F_x$;
- $\Phi\big(f(x), y\big) = y$ for all $y \in F_x$.

In other words, the map $\Phi$ is a local trivialization that restricts to the inclusion of $F_x$ over the point $f(x)$.

*Proof.* If $U$ is a chart on $N$ that contains $f(x)$, we may restrict our view to the map $f : f^{-1}(U) \to U$ without loss of generality. As such, we may assume that $N = \mathbb{R}^n$ and $f(x) = 0$. For each $i = 1, \dots, n$, let $X_i$ denote the horizontal lift of $\partial/\partial x_i$. Each $X_i$ admits a maximal flow domain $\mathscr{D}_i \subseteq M \times \mathbb{R}$ and a flow $\psi_i : \mathscr{D}_i \to M$. We will inductively define neighborhoods $U_i \subseteq \mathbb{R}^i$ of the origin and immersions $\Phi_i : U_i \times F_x \to M$ such that:

- $f \circ \Phi_i(c, y) = \jmath_i(c)$ for all $c \in U_i$ and $y \in F_x$, where $\jmath_i : \mathbb{R}^i \to \mathbb{R}^n$ is inclusion in the first $i$ coordinates;
- $\Phi_i\big(0_i, y\big) = y$ for all $y \in F_x$, where $0_i$ denotes the origin in $\mathbb{R}^i$.

Then setting $U = U_n$ and $\Phi = \Phi_n$ will complete the proof. Indeed, the desired properties of $\Phi$ follow immediately from the corresponding propeties of $\Phi_n$. In particular, the fact that $\Phi$ is an open embedding follows from the fact that $\Phi_n$ is an immersion and the following "rank-nullity" calculation:

$$\dim(U \times F_x) = \dim U + \dim F_x = \dim N + \dim f^{-1}(x) = \dim M.$$

First, we define $\Phi_0 : F_x \to M$ to be the inclusion map. This clearly satisfies all of the desired conditions. Now fix $i < n$ and suppose that we have constructed $U_i$ and $\Phi_i$ with the desired properties. Define a map

$$\iota : U_i \times \mathbb{R} \times F_x \to M \times \mathbb{R} \quad \text{by} \quad \iota(t_1, \dots, t_n, t, y) = \big(\Phi_i(t_1, \dots, t_n, y), t\big).$$

If we write $\mathscr{E} = \iota^{-1}(\mathscr{D}_{i+1})$, then $\mathscr{E}$ is open in $U_i \times \mathbb{R} \times F_x$ and thus in $\mathbb{R}^{i+1} \times F_x$. For any $y \in F_x$, we have

$$\iota(0_i, 0, y) = \big(\Phi_i(0_i, y), 0\big) = (y, 0) \in \mathscr{D}_{i+1}$$

by our second assumption on $\Phi_i$ and the definition of a flow domain, so $\mathscr{E}$ is a neighborhood of $\{0_{i+1}\} \times F_x$. Since $\iota(\mathscr{E}) \subseteq \mathscr{D}_{i+1}$, we let the map $\Phi_{i+1} : \mathscr{E} \to M$ be defined by $\Phi_{i+1} = \psi_{i+1} \circ \iota$. More explicitly, we have

$$\Phi_{i+1}(t_1, \dots, t_i, t_{i+1}, y) = \psi_{i+1}\big(\Phi_i(t_1, \dots, t_i, y), t_{i+1}\big).$$

Because the vector fields $X_{i+1}$ and $\partial/\partial x_{i+1}$ are $f$-related, we can observe that $f \circ \psi_{i+1}(z,t) = f(z) + te_{i+1}$ for any $(z,t) \in \mathscr{D}_{i+1}$. Combining this with our first assumption on $\Phi_i$, we get

$$\begin{aligned} f \circ \Phi_{i+1}(t_1, \ldots, t_i, t_{i+1}, y) &= f \circ \psi_{i+1}\big(\Phi_i(t_1, \ldots, t_i, y), t_{i+1}\big) \\ &= f \circ \Phi_i(t_1, \ldots, t_i, y) + t_{i+1} e_{i+1} \\ &= \jmath_i(t_1, \ldots, t_i) + t_{i+1} e_{i+1} \\ &= \jmath_{i+1}(t_1, \ldots, t_i, t_{i+1}). \end{aligned}$$

Since $\jmath_{i+1}$ is obviously an immersion, this equality implies that $D\Phi_{i+1}$ cannot annihilate any vectors tangent to a slice $\mathscr{E} \cap \big(\mathbb{R}^{i+1} \times \{y\}\big)$ (where $y \in F_x$ is arbitrary). Using the second assumption on $\Phi_i$, we also have

$$\Phi_{i+1}(0_{i+1}, y) = \psi_{i+1}\big(\Phi_i(0_i, y), 0\big) = \psi_{i+1}(y, 0) = y.$$

In particular, this fact implies that $D\Phi_{i+1}$ cannot annihilate any vectors tangent to the slice $\{0_{i+1}\} \times F_x$. Combining this with the above observation, we now see that $D\Phi_{i+1}$ is injective at any point in $\{0_{i+1}\} \times F_x$. The rank of a smooth map is lower semi-continuous, so there is an open set $W \subseteq \mathscr{E}$ containing $\{0_{i+1}\} \times F_x$ on which $\Phi_{i+1}$ is an immersion. Since $F_x$ is compact, there is some neighborhood $U_{i+1} \subseteq \mathbb{R}^{i+1}$ of the origin with $U_{i+1} \times F_x \subseteq W$.[1] This completes the inductive step and thus completes the proof of the theorem. □

**Corollary.** The set $\{x \in M : F_x \text{ is compact}\}$ is open in $M$.

*Proof.* Suppose that $F_x$ is compact and let $\Phi : U \times F_x \to M$ be as in the above theorem. Then $\Phi(U \times F_x)$ is open in $M$ and we will show that $F_y$ is compact for any $y \in \Phi(U \times F_x)$. Suppose $y = \Phi(u, z)$ with $u \in U$ and $z \in F_x$. Then $S = \Phi\big(\{u\} \times F_x\big)$ is a compact, connected submanifold of $M$ satisfying $y \in S \subseteq f^{-1}(u)$. Because $f$ is a submersion, all of its fibers have the same dimension and thus

$$\dim S = \dim F_x = \dim f^{-1}\big(f(x)\big) = \dim f^{-1}(u).$$

Since $S \subseteq f^{-1}(u)$ is a full-dimensional submanifold, it is open in $f^{-1}(u)$. But we also know that $S$ is compact and connected, so it must be a connected component of $f^{-1}(u) = f^{-1}\big(f(y)\big)$. But $y \in S$, so we get $F_y = S$, from which we can immediately conclude that $F_y$ is compact. □

[1] If $X$ is compact and $p \in Y$, then any open set $W \subseteq X \times Y$ containing $X \times \{p\}$ must also contain a set of the form $X \times U$, where $U \subseteq Y$ is an open set containing $p$. The proof is a straightforward application of the common "union/intersection trick" that is applied to prove various point-set-topological properties involving compactness (e.g. compact Hausdorff $\Longrightarrow$ regular).