Extremal Values of Pi

Nikhil Henry Bukowski Sahoo

Abstract. We discuss the classical results of Stanisław Gołąb, on the values of pi in arbitrary normed planes, including a classification of extremal values. We reprove a result of J. Duncan, D. Luecking, and C. McGregor, which states that any norm with quarter-turn symmetry has pi-value at least π. We also show that a norm is Euclidean if and only if it has quarter-turn symmetry in some basis and pi-value π.

1. INTRODUCTION. In 1932, Polish geometer Stanisław Gołąb posed and solved an interesting problem: if we generalize the notion of pi $=$ circumference$/$diameter to unit circles of arbitrary norms on the plane, then the possible pi-values comprise the interval $[3, 4]$. The article [**3**] provides a wonderful exposition and proves some related results, including a new theorem showing that norms on $\mathbb{R}^2$ with quarter-turn symmetry only attain pi-values in the interval $[\pi, 4]$. We will recount these results from scratch, while also classifying the norms that achieve extreme pi-values. For the extreme values of 3 and 4, the classification was given by Gołąb in his original paper. For the extreme value of π, with quarter-turn symmetry, the classification is—to the best of my knowledge—new. We will then relate this final result to a fundamental question of Minkowski geometry: what conditions are both necessary and sufficient for a normed space to be Euclidean?

A *norm* X on $\mathbb{R}^n$ is given by a function $||\cdot||_X : \mathbb{R}^n \to [0, \infty)$ such that:

- $||v||_X = 0$ if and only if $v = 0$;
- $||cv||_X = |c| \cdot ||v||_X$ for all $c \in \mathbb{R}$;
- and $||u + v||_X \leq ||u||_X + ||v||_X$.

Typical examples include the ℓ^p norms on $\mathbb{R}^n$ for all $p \geq 1$:

$$||v||_p = \sqrt[p]{|v_1|^p + \cdots + |v_n|^p}. \tag{1}$$

Taking the limit of (1) as $p \to \infty$ gives the ℓ^∞ norm: $||v||_\infty = \max\{|v_1|, \ldots, |v_n|\}$. The most common norm is ℓ^2, for which (1) is essentially just the "distance formula." For any norm X on $\mathbb{R}^n$, we may define the unit ball and its boundary, the unit sphere:

$$B_X = \{v \in \mathbb{R}^n : ||v||_X \leq 1\} \qquad \text{and} \qquad \partial B_X = \{v \in \mathbb{R}^n : ||v||_X = 1\}.$$

Then B_X is compact and convex, with $-B_X = B_X$ and $0 \in B_X^\circ$ (the interior of B_X). Conversely, for any $B \subset \mathbb{R}^n$ with these properties and any $v \in \mathbb{R}^n$, we may define

$$||v||_X = \frac{1}{\sup\{a \in \mathbb{R} : av \in B\}}.$$

This is the unique norm X with $B_X = B$. This bijection between norms and certain convex sets gives the study of normed spaces a geometric flavor: the shape of the unit ball regulates properties of the norm. For example, we will see that the eponymous "extremal values of pi" occur if and only if B_X is an affine regular hexagon, ellipse, or parallelogram.

Before diving into anything technical, we can already observe a host of pi-values:

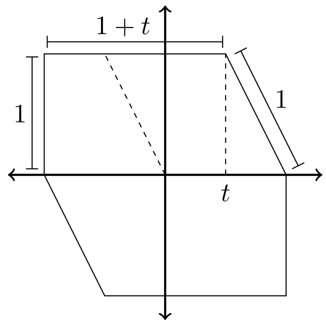

Figure 1. A norm with pi-value $3 + t$, for any $t \in [0, 1]$.

Example 1.1. Fix $t \in [0, 1]$ and let X be the norm on $\mathbb{R}^2$ whose unit disk is

$$B_X = \text{conv}\{e_1, te_1 + e_2, e_2 - e_1, -e_1, -te_1 - e_2, e_1 - e_2\}.^1$$

This is shown in Figure 1. Note that B_X is a hexagon, except in the case of $t = 1$, when it degenerates into a square. Since $e_1, e_2, e_2 - (1 - t)e_1 \in \partial B_X$, we see that these vectors have norm 1 under X. Hence, the upper half of the unit circle has length

$$\begin{aligned} &||-e_1 - (e_2 - e_1)||_X + ||(e_2 - e_1) - (te_1 + e_2)||_X + ||(te_1 + e_2) - e_1||_X \\ &\qquad = ||e_2||_X + (1 + t)||e_1||_X + ||e_2 - (1 - t)e_1||_X = 3 + t. \end{aligned}$$

The bottom half of the unit circle is analogous, so we see that ∂B_X has length $6 + 2t$. Notably, this length is measured in terms of the norm X, so it is sometimes called the *self-circumference* of X. To find the value of pi for X, we divide this circumference by the length 2 of the diameter, yielding $3 + t$. As t ranges over $[0, 1]$, this indeed shows that any real number in the interval $[3, 4]$ is the value of pi in some normed plane.

In what follows, our first goal will be to understand the length of any curve in terms of any norm, so that we may study the "self-circumference" of arbitrary norms on $\mathbb{R}^2$.

2. ARC-LENGTH IN TERMS OF A NORM. A *metric* on a set A is a symmetric function $d : A \times A \to [0, \infty)$ such that

- $d(u, v) = 0$ if and only if $u = v$;
- and $d(u, w) \leq d(u, v) + d(v, w)$.

A norm X on $\mathbb{R}^n$ induces a metric $d_X(u, v) = ||u - v||_X$ on $\mathbb{R}^n$, with the additional properties $d_X(u + w, v + w) = d_X(u, v)$ and $d(au, av) = a \cdot d(u, v)$ for any $a > 0$ and $u, v, w \in \mathbb{R}^n$. Conversely, any metric on $\mathbb{R}^n$ with these properties defines a norm.

A metric d on a set A defines some notion of "shortest distance" between two points, but it can also be useful to consider the length travelled along more circuitous paths. For any $u, v \in A$, a *path* from u to v is any function $\varphi : [a, b] \to A$ with $\varphi(a) = u$ and $\varphi(b) = v$. If $u = v$, then we call φ a *loop* with basepoint u. The length of φ is

$$\text{len}_d\varphi = \sup\left\{\sum_{i=1}^{n} d\big(\varphi(t_{i-1}), \varphi(t_i)\big) : a = t_0 \leq \cdots \leq t_n = b \text{ and } n \in \mathbb{N}\right\}.$$

[1]We write conv(A) to denote the convex hull of A, which here is simply the polygon with these vertices.

We call the sequence $\varphi(t_0), \ldots, \varphi(t_n)$ a *partition* of the path φ. The resulting length always takes a well-defined value in $[0, \infty]$. First, we note some elementary properties. As the results are intuitively plausible, the proofs are left to the interested reader.

(a) For any paths $\varphi_1 : [a, b] \to A$ and $\varphi_2 : [b, c] \to A$ such that $\varphi_1(b) = \varphi_2(b)$, we define their *concatenation* $\varphi_1 \bullet \varphi_2 : [a, c] \to A$ as

$$(\varphi_1 \bullet \varphi_2)(t) = \begin{cases} \varphi_1(t), & a \le t \le b. \\ \varphi_2(t), & b \le t \le c. \end{cases}$$

In the arithmetic of $[0, \infty]$, we then have $\text{len}_d(\varphi_1 \bullet \varphi_2) = \text{len}_d(\varphi_1) + \text{len}_d(\varphi_2)$.

(b) Given a loop $\varphi : [a, c] \to A$ and any $b \in [a, c]$, we may define

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & b \le t \le c; \\ \varphi(t - c + a), & c \le t \le b + c - a. \end{cases}$$

This loop $\tilde{\varphi} : [b, b + c - a] \to A$ is essentially just φ with a shifted basepoint. Using (a), we see that $\text{len}_d\tilde{\varphi} = \text{len}_d\varphi$, so the basepoint does not matter when measuring length. As such, we view loops as functions $S^1 \to A$ from the circle, which possess a well-defined notion of length (measured from any basepoint).

(c) If $\varphi : [c, d] \to A$ is a path and $f : [a, b] \to [c, d]$ is monotonic and surjective, then $\text{len}_d(\varphi \circ f) = \text{len}_d\varphi$. Thus if φ is an injective, continuous path or loop, then $\text{len}_d\varphi$ depends only on the image of φ. Additionally, given any two curves, we can translate their domains so that the concatenation in (a) is well-defined; the resulting length will be independent of how this is done. Whenever we write a concatenation of curves, such a reparametrization will be implicitly assumed.

(d) For any $u, v \in \mathbb{R}^n$, we use the notation $[u, v] = \{(1 - t)u + tv : 0 \le t \le 1\}$ and $(u, v) = \{(1 - t)u + tv : 0 < t < 1\}$ for the closed and open segments from u to v. For any sequence $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$, we have a polygonal path

$$[x_0, x_1, \ldots, x_n] = [x_0, x_1] \bullet \cdots \bullet [x_{n-1}, x_n].$$

For any norm X on $\mathbb{R}^n$, the length in terms of d_X of such a path is

$$\text{len}_X[x_0, x_1, \ldots, x_n] = d_X(x_0, x_1) + \cdots + d_X(x_{n-1}, x_n).$$

By additivity of lengths in (a), it suffices to prove that $\text{len}_X[u, v] = ||u - v||_X$.

(e) Suppose that X and Y are norms on $\mathbb{R}^n$ with $B_X \subset B_Y$. Then $||v||_Y \le ||v||_X$ for any $v \in \mathbb{R}^n$, and thus $\text{len}_Y\varphi \le \text{len}_X\varphi$ for any path $\varphi : [a, b] \to \mathbb{R}^n$.

We write $\mathcal{M}$ for the set of all norms on $\mathbb{R}^2$, and we will only be considering lengths in terms of these norms. It can also be useful to identify $\mathbb{R}^2 = \mathbb{C}$ and use some notation befitting complex numbers. In particular, a counterclockwise quarter-turn is $z \mapsto iz$, and the ℓ^2 norm is just the absolute value $|z| = ||z||_2$. We will speak of angles in terms the function $\arg : \mathbb{C} - 0 \to S^1$. Angle measure will always be considered in radians.

Convex paths. In general, curves need not have finite length with respect to a norm, even if they are continuous and injective.[2] However, we are interested in curves that form a portion of the boundary of a convex set in $\mathbb{R}^2$. Below, we will prove a useful comparison lemma; in particular, this will imply that all such curves have finite length.

[2] A typical example is the Koch snowflake. In general, a continuous, injective curve or loop has finite length if and only if its Hausdorff dimension is 1 (or 0, in the case when the "curve" is just a single point) [**4**].

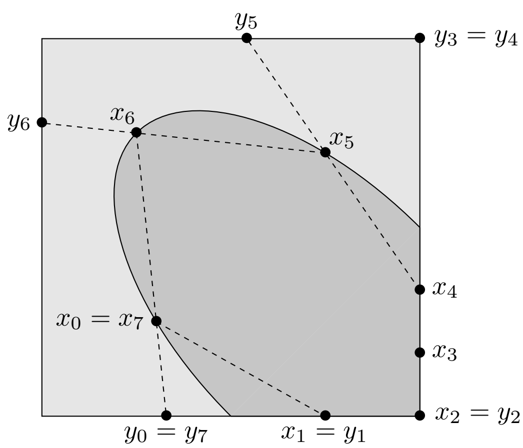

Figure 2. Comparing the lengths of curves bounding convex regions.

If $B \subset \mathbb{R}^n$ is a compact, convex set with $B^\circ \neq \emptyset$, we will call B a *convex body*. Then a set $B \subset \mathbb{R}^n$ is the unit ball of a norm on $\mathbb{R}^n$ if and only if B is a convex body and symmetric about the origin. If $B \subset \mathbb{R}^2$ is a convex body, then ∂B is a continuous, injective loop: choose any $x \in B^\circ$ and define $\varphi : \partial B \to S^1$ by $\varphi(y) = \arg(y - x)$. The resulting length depends only on the set ∂B, so any $x \in B^\circ$ works equally well. We now compare the lengths of ∂B for various such B.

Lemma 2.1. *If $B_1, B_2 \subset \mathbb{R}^2$ are any two convex bodies with $B_1 \subset B_2$, then we have $\mathrm{len}_X(\partial B_1) \leq \mathrm{len}_X(\partial B_2)$ for any norm $X \in \mathcal{M}$.*

Proof. We closely follow the proof of Lemma 1(a) in [**3**]. Let $x_0, \ldots, x_n$ be a partition of ∂B_1. Since ∂B_1 is a loop, we have $x_0 = x_n$. For each $i = 1, \ldots, n$, we define

$$y_i = x_{i-1} + \sup\{t \in [0, \infty) : x_{i-1} + t(x_i - x_{i-1}) \in B_2\}(x_i - x_{i-1}).$$

This is the furthest point along the ray $R = \{x_{i-1} + t(x_i - x_{i-1}) \in \mathbb{R}^2 : t \in [0, \infty)\}$ that is also contained in B_2. In particular, we see that $y_i \in \partial B_2$. (Often, but not always, y_i is the unique point in $R \cap \partial B_2$.) We also set $y_0 = y_n$, so the sequence $y_0, \ldots, y_n$ is a partition of ∂B_2. This is illustrated in Figure 2. Because $x_i \in [x_{i-1}, y_i]$, we have

$$d_X(x_{i-1}, x_i) + d_X(x_i, y_i) = d_X(x_{i-1}, y_i) \leq d_X(x_{i-1}, y_{i-1}) + d_X(y_{i-1}, y_i)$$

for all $i = 1, \ldots, n$, by the triangle inequality. Summing these inequalities, we have

$$\sum_{i=1}^{n} d_X(x_{i-1}, x_i) + \sum_{i=1}^{n} d_X(x_i, y_i) \leq \sum_{i=1}^{n} d_X(x_{i-1}, y_{i-1}) + \sum_{i=1}^{n} d_X(y_{i-1}, y_i).$$

Since $x_0 = x_n$ and $y_0 = y_n$, the middle two sums are equal and therefore

$$\sum_{i=1}^{n} d_X(x_{i-1}, x_i) \leq \sum_{i=1}^{n} d_X(y_{i-1}, y_i) \leq \mathrm{len}_X(\partial B_2).$$

The partition $x_0, \ldots, x_n$ was arbitrary, so this gives $\mathrm{len}_X(\partial B_1) \leq \mathrm{len}_X(\partial B_2)$. ■

If $B \subset \mathbb{R}^2$ is a convex body, then we have $B \subset [-a, a]^2$ for large enough $a > 0$. Since polygonal curves clearly have finite length, Lemma 2.1 gives $\text{len}_X(\partial B) < \infty$. More generally, we make the following definition:

Definition 2.2. We say that a path φ from u to v is *convex* if $\varphi = [u, v]$ or if $\varphi \bullet [v, u]$ is the boundary of a convex body. We write $\varphi_1 \prec \varphi_2$ if φ_1 and φ_2 are both convex paths from u to v such that $\text{conv}(\varphi_1) \subset \text{conv}(\varphi_2)$.

For any convex path φ and $X \in \mathcal{M}$, we have $\text{len}_X\varphi \leq \text{len}_X\big(\varphi \bullet [v, u]\big) < \infty$. Thus all convex paths have finite length. We will need some further results concerning convex paths and their lengths. A different proof of (a) can be found in [**12**, §§4.3–4.4].

Lemma 2.3. (a) *If $\varphi_1 \prec \varphi_2$, then $\text{len}_X\varphi_1 \leq \text{len}_X\varphi_2$ for any norm $X \in \mathcal{M}$.*

(b) *If φ is a convex path between two distinct points $u, v \in \mathbb{R}^2$ and $\varphi \cap (u, v) \neq \emptyset$, then $\varphi = [u, v]$.*

(c) *If $B \subset \mathbb{R}^2$ is convex, with $x \in B$ and $y \in B^\circ$, then $(x, y) \subset B^\circ$.*

(d) *If $B \subset \mathbb{R}^2$ is convex and $x, z \in B$, then either $(x, z) \subset \partial B$ or $(x, z) \subset B^\circ$.*

(e) *If $B \subset \mathbb{R}^2$ is a convex body, then any path along its boundary (i.e., a nonempty, closed, connected subset of ∂B) is a convex path.*

(f) *If $\varphi : [a, b] \to \mathbb{R}^2$ is a convex path and $[c, d] \subset [a, b]$, then $\varphi|_{[c,d]}$ is also convex.*

(g) *Let φ be a convex path and suppose that the points $p, q, r \in \varphi$ occur in this order along the path φ. If $q \in (p, r)$, then $[p, r] \subset \varphi$.*

Proof. (a) We leave the case when $\varphi_1 = [u, v]$ to the reader. If φ_2 is a line segment, then $\varphi_1 \prec \varphi_2$ implies that φ_1 is as well. Hence, for $i = 1$ or 2, we see that $\text{conv}(\varphi_i)$ is a convex body bounded by $\partial\,\text{conv}(\varphi_i) = \varphi_i \bullet [v, u]$. Thus, Lemma 2.1 gives

$$\begin{aligned}\text{len}_X\varphi_2 + d_X(v, u) &= \text{len}_X\big(\varphi_2 \bullet [v, u]\big) \\ &\geq \text{len}_X\big(\varphi_1 \bullet [v, u]\big) = \text{len}_X\varphi_1 + d_X(v, u),\end{aligned}$$

since $\text{conv}(\varphi_1) \subset \text{conv}(\varphi_2)$. The desired inequality follows immediately.

(b) If $\varphi \bullet [v, u]$ is the boundary of a convex body, then φ and $[v, u]$ only intersect at u and v, so $\varphi \cap (u, v) = \emptyset$. This contradicts our initial assumption, so $\varphi = [u, v]$.

(c) We may suppose that $x \neq y$, in which case $v = i(x - y)$ is a nonzero vector orthogonal to $x - y$. Since $y \in B^\circ$, we can find $\epsilon > 0$ with $y \pm \epsilon v \in B$. The triangle $T = \text{conv}(x, y + \epsilon v, y - \epsilon v\}$ is contained in B, by convexity. But y lies in the interior of the edge of T opposite the vertex x, which clearly gives $(x, y) \subset T^\circ \subset B^\circ$.

(d) Suppose $y \in (x, z) \cap B^\circ$. Using (c) twice gives $(x, y) \subset B^\circ$ and $(z, y) \subset B^\circ$, and thus $(x, z) \subset B^\circ$. On the other hand, if no such y exists, then $(x, z) \subset \partial B$.

(e) For brevity, we will only sketch this proof. Let φ be a path from u to v along ∂B. By (d), either $(u, v) \subset B^\circ$ or $(u, v) \subset \partial B$. But if $(u, v) \subset \partial B$, then either $\varphi = [u, v]$ or $\varphi \bullet [v, u] = \partial B$, so φ is convex. Thus we suppose that $(u, v) \subset B^\circ$. Then the line ℓ through u and v only intersects ∂B in these two points, and because φ is connected, it must lie in one of the closed half-planes cut out by ℓ. If H denotes this half-plane, then since H is convex and closed, we can see that $B \cap H$ is a convex body such that $\partial(B \cap H) = \varphi \bullet [v, u]$. This shows that φ is a convex path in any of the above cases.

(f) Let $\varphi(a) = u$ and $\varphi(b) = v$. If $\varphi = [u, v]$, then $\varphi|_{[c,d]}$ is clearly a line segment. Otherwise, $\varphi \bullet [v, u]$ is the boundary of a convex body B. Then $\varphi|_{[c,d]}$ is a path along the boundary ∂B, so φ is a convex path by (e).

(g) Let ψ be the portion of φ between p and r. Then $q \in \psi \cap (p, r)$ by assumption, and (f) shows that ψ is a convex path from p to r, so (b) gives $[p, r] = \psi \subset \varphi$. ∎

3. WHAT VALUES DOES PI TAKE? With the above notion of length, we can now define the promised generalization of pi for any norm on $\mathbb{R}^2$.

Definition 3.1. For any norm $X \in \mathcal{M}$, define $\varpi(X) = \text{len}_X(\partial B_X)/2$. This symbol ϖ was historically used as a cursive π. The characters mean two different things to us: we write ϖ for "pi in terms of a norm" and $\pi = 3.14159\ldots$ for the classic constant.

The results of the previous section show us that $\varpi(X) \in (0, \infty)$ for any norm X. Since $-B_X = B_X$, we see that the loop ∂B_X has half-turn symmetry, so we can also calculate $\varpi(X)$ as the length of the intersection of ∂B_X with the upper half-plane.

In this section, we will eventually show that the image is $\varpi(\mathcal{M}) = [3, 4]$. Already, the polygonal norms treated in Example 1.1 show that $[3, 4] \subset \varpi(\mathcal{M})$.

Notice that $\varpi(X)$ depends on X in two ways: the unit circle ∂B_X is defined by X, but we also use X to measure length. These two dependencies are precisely balanced, in a way that we will now make precise.

Definition 3.2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. Given any norm X on $\mathbb{R}^n$, the *push-forward* norm TX is defined by $||v||_{TX} = ||T^{-1}v||_X$. If norms X_1 and X_2 satisfy $TX_1 = X_2$ for some isomorphism T, we will call them *linearly equivalent*. Similarly, we say that two sets $A_1, A_2 \subset \mathbb{R}^n$ are linearly equivalent if $T(A_1) = A_2$ for some linear isomorphism T.

First, note that $B_{TX} = T(B_X)$ for any norm X on $\mathbb{R}^n$ and linear isomorphism T. Thus we can see that norms X_1 and X_2 are linearly equivalent if and only if their unit balls B_{X_1} and B_{X_2} are linearly equivalent. This fact will be of frequent use below.

Fix an isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ and a norm X on $\mathbb{R}^n$. For $u, v \in \mathbb{R}^n$, we have

$$d_X(u, v) = ||u - v||_X = ||Tu - Tv||_{TX} = d_{TX}(Tu, Tv).$$

Now consider an arbitrary curve $\varphi : [a, b] \to \mathbb{R}^n$. If $x_0, \ldots, x_n$ is a partition of φ, then $T(x_0), \ldots, T(x_n)$ is a partition of $T \circ \varphi$ and we have

$$\sum_{i=1}^{n} d_X(x_{i-1}, x_i) = \sum_{i=1}^{n} d_{TX}\big(T(x_{i-1}), T(x_i)\big) \leq \text{len}_{TX}(T \circ \varphi).$$

This proves that $\text{len}_X\varphi \leq \text{len}_{TX}(T \circ \varphi)$. Replacing T by T^{-1} gives

$$\text{len}_{TX}(T \circ \varphi) \leq \text{len}_{T^{-1}TX}(T^{-1} \circ T \circ \varphi) = \text{len}_X\varphi.$$

Thus $\text{len}_X\varphi = \text{len}_{TX}(T \circ \varphi)$. Note that $\partial B_{TX} = T(\partial B_X)$, so when $n = 2$ we have

$$\varpi(X) = \text{len}_X(\partial B_X)/2 = \text{len}_{TX}(\partial B_{TX})/2 = \varpi(TX).$$

Lemma 3.3. *The map $\varpi : \mathcal{M} \to (0, \infty)$ is constant on linear equivalence classes.*

There are three linear equivalence classes of particular importance:

(a) The constant π is defined as $\pi = \varpi(\ell^2)$. Since ∂B_{ℓ^2} is the classical unit circle, a norm $X \in \mathcal{M}$ is linearly equivalent to ℓ^2 if and only if B_X is a filled ellipse. Therefore, all ellipses (centered at the origin) yield $\varpi = \pi$.

(b) Taking $t = 1$ in Example 1.1, notice that the unit ball is precisely

$$B_{\ell^\infty} = \{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\},$$

so our calculation in Example 1.1 gives $\varpi(\ell^\infty) = 4$. This unit circle is a square, so $X \in \mathcal{M}$ is linearly equivalent to ℓ^∞ if and only if B_X is a parallelogram. Therefore, all parallelograms (centered at the origin) yield $\varpi = 4$. In particular, since B_{ℓ^1} is the square with vertices $\{e_1, -e_1, e_2, -e_2\}$, we have $\varpi(\ell^1) = 4$.

(c) We will say that $B \subset \mathbb{R}^2$ is a *linearly regular hexagon* if B is linearly equivalent to a regular hexagon centered at the origin. This is equivalent to requiring that

$$B = \operatorname{conv}\{u, v, v - u, -u, -v, u - v\}$$

for some linearly independent $u, v \in \mathbb{R}^2$. (This is an actual regular hexagon when $|u|^2 = |v|^2 = 2\langle u, v\rangle$.) Taking $t = 0$ in Example 1.1 gives us a norm X, where B_X is a linearly regular hexagon (take $u = e_1$ and $v = e_2$). We calculated that $\varpi(X) = 3$, so we can see that all linearly regular hexagons yield $\varpi = 3$.

In what follows, we will show that $\varpi(\mathcal{M}) = [3, 4]$ and that (b) and (c) characterize the extremal cases. More specifically, $\varpi(X) = 4$ if and only B_X is a parallelogram (centered at the origin), and $\varpi(X) = 3$ if and only if B_X is a linearly regular hexagon. For essentially all of this, we will closely follow Schäffer's approach in [**11**].[3]

Circumscribed parallelograms. We will now address the upper bound $\varpi \le 4$. First, we prove that any norm on $\mathbb{R}^2$ can be put into a particular "normalized" form.

Lemma 3.4. *For any $X \in \mathcal{M}$, there exists some isomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $||e_1||_{TX} = ||e_2||_{TX} = 1$ and $||(x, y)||_{TX} \ge \max\big(|x|, |y|\big)$.*

Proof. This proof differs from [**3, 11**] and more closely follows [**12**, Theorem 3.2.1]. We view $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$ so that we may consider cross products. Since B_X is compact, we can find two vectors $u, v \in B_X$ that maximize $||u \times v||_2$. Now define

$$P = \{su + tv : |s| \le 1 \text{ and } |t| \le 1\}.$$

Suppose that $su + tv \in B_X$. Then we have $|s| \le 1$, because

$$|s| \cdot ||u \times v||_2 = ||su \times v||_2 = ||(su + tv) \times v||_2 \le ||u \times v||_2,$$

and we can similarly show that $|t| \le 1$. This proves that $B_X \subset P$.

Since $u, v \in \partial P$, we must also have $u, v \in \partial B_X$. Therefore $||u||_X = ||v||_X = 1$. If $a = \max\big(|s|, |t|\big)$ for some $s, t \in \mathbb{R}$, then $aB_X \subset aP$ and thus $||su + tv||_X \ge a$. Hence, if we set $T(u) = e_1$ and $T(v) = e_2$, we get the desired isomorphism T. ■

Using this result, the proof of $\varpi \le 4$ is almost immediate, although a little more work will be needed to classify all $X \in \mathcal{M}$ satifying $\varpi(X) = 4$.

Theorem 3.5. *For any norm $X \in \mathcal{M}$, we have $\varpi(X) \le 4$. Moreover, we have $\varpi(X) = 4$ if and only if B_X is a parallelogram (centered at the origin).*

Proof. Using Lemma 3.4, we may assume that $||(x, y)||_X \ge \max\big(|x|, |y|\big)$ and that $||e_1||_X = ||e_2||_X = 1$, because ϖ and the property of being a parallelogram are both preserved under linear equivalence. Then $B_X \subset B_{\ell^\infty}$ and therefore Lemma 2.1 gives

$$2\varpi(X) = \operatorname{len}_X(\partial B_X) \le \operatorname{len}_X(\partial B_{\ell^\infty}) = 4\big(||e_1||_X + ||e_2||_X\big) = 8.$$

[3] The original paper [**7**] is only available as a physical copy in a few libraries, making it fairly inaccessible, particularly during a pandemic. As such, most authors reference [**7**] by way of a French-language summary, which has had the side-effect of dissociating the paper (and even the journal name) from its original language. While [**11**] does not credit Gołąb with these results, it reproduces all the proofs (making no claim to originality) and sets forth fascinating new avenues of study by generalizing ϖ to higher dimensions in multiple ways.

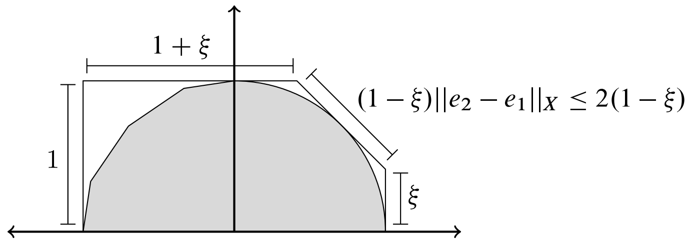

Figure 3. Classifying the case when $\varpi = 4$.

Now suppose that $\varpi(X) = 4$. If $e_1 + e_2, e_1 - e_2, -e_1 - e_2, e_2 - e_1 \in B_X$, then

$$B_{\ell^\infty} = \operatorname{conv}\{e_1 + e_2, e_1 - e_2, -e_1 - e_2, e_2 - e_1\} \subset B_X \subset B_{\ell^\infty}.$$

Then $X = \ell^\infty$ and B_X is thus a square. Hence, we may assume that one of these points is not in B_X; after a rotation, we may assume that $e_1 + e_2 \notin B_X$. We define

$$\xi = \max\{x + y - 1 : (x, y) \in B_X\}.$$

Note that ξ exists by the compactness of B_X and $0 \leq \xi < 1$ because $e_1 + e_2 \notin B_X$. Let φ be the portion of ∂B_X in the upper half-plane, a path from e_1 to $-e_1$. Then

$$\varphi \prec [e_1, e_1 + \xi e_2, \xi e_1 + e_2, e_2 - e_1, -e_1],$$

as illustrated in Figure 3. Therefore, Lemma 2.3(a) gives

$$\begin{aligned} 4 = \operatorname{len}_X \varphi &\leq \operatorname{len}_X [e_1, e_1 + \xi e_2, \xi e_1 + e_2, e_2 - e_1, -e_1] \\ &= ||\xi e_2||_X + ||(1-\xi)(e_2 - e_1)||_X + ||(1+\xi)e_1||_X + ||e_2||_X \\ &= 2 + 2\xi + (1-\xi)||e_2 - e_1||_X \leq 2(1+\xi) + 2(1-\xi) = 4, \end{aligned}$$

since $||e_2 - e_1||_X \leq ||e_2||_X + ||e_1||_X = 2$. We then must have equality throughout; in particular, the last inequality becomes $||e_2 - e_1||_X = 2$ (since $1 - \xi > 0$). Hence, $\frac{1}{2}(e_2 - e_1) \in \partial B_X$ and so $[e_2, -e_1] \subset \partial B_X$ by Lemma 2.3(g). Then $e_2 - e_1 \notin B_X$, so we may repeat this argument with e_1 negated, to show that $[e_2, e_1] \subset \partial B_X$ as well. Therefore $\varphi = [e_1, e_2, -e_1]$, which implies that $X = \ell^1$ and B_X is thus a square. ■

Inscribed hexagons. We now prove that $\varpi \geq 3$. In classifying the case when $\varpi = 3$, we need some notions from convex geometry, of which we assume no prior knowledge. To the more experienced reader, some of the choices in presenting this background material may seem a little odd, but they have all been chosen to best fit what follows.

Definition 3.6. Let $B \subset \mathbb{R}^2$ be a convex set. If $F \subset B$ is nonempty, convex, and

$$x, y \in B \text{ and } (x, y) \cap F \neq \emptyset \quad \Longrightarrow \quad x, y \in F,$$

we call F a *face* of B. We say that $p \in B$ is an *extreme point* if $\{p\}$ is a face, i.e.,

$$x, y \in B \text{ and } p \in (x, y) \quad \Longrightarrow \quad x = y = p.$$

If $\ell \subset \mathbb{R}^2$ is any line where B is entirely in one of the closed half-planes cut out by ℓ, and $q \in \ell \cap B$, we say that ℓ *supports* B at the point q. If the point q is left unspecified, it is still implied that a supporting line must intersect B.

We will use supporting lines to prove that convex bodies in $\mathbb{R}^2$ always contain extreme points. But first, we leave the following facts for the reader to verify:

(a) If F is a face of B, then $E \subset F$ is a face of F if and only if E is a face of B.
(b) If $F \subset B$ is a proper face of B (i.e., $F \neq B$), then $F \subset \partial B$.
(c) For any $p, q \in \mathbb{R}^2$, the extreme points of the line segment $[p, q]$ are p and q.

Lemma 3.7. *Let $B \subset \mathbb{R}^2$ be a convex body and let $\ell \subset \mathbb{R}^2$ be any line.*

(a) *If ℓ is a supporting line for B, then $\ell \cap B$ is a face of B.*
(b) *The line ℓ supports B at q if and only if $q \in \ell \cap B \subset \partial B$.*
(c) *If there exist distinct $u, v \in \ell \cap B$ with $[u, v] \subset \partial B$, then $\ell \cap B$ is a face of B and a closed line segment.*
(d) *B has an extreme point.*

Proof. (a) As an intersection of two convex sets, the set $F = \ell \cap B$ is also convex. Because ℓ is a supporting line for B, this intersection is also nonempty. Suppose that $x, y \in B$ satisfy $(x, y) \cap F \neq \emptyset$. Then the open segment (x, y) intersects ℓ, so either $x, y \in \ell$ or x and y lie strictly on opposite sides of ℓ. The latter is impossible, since B is contained in one of the closed half-planes cut out by ℓ. Therefore, we have $x, y \in \ell$ and thus $x, y \in F$. This proves that F is a face of B.

Since B is compact and convex, notice that $F = \ell \cap B$ is a compact, convex subset of the line ℓ. Thus $F = [u, v]$ for some $u, v \in \ell$. We will use this in (c) and (d) below.

(b) First, suppose that ℓ supports B at q. This means that $q \in \ell \cap B$ and $B \subset H$, where H is a closed half-plane cut out by ℓ. Then $B - \partial B = B^\circ \subset H^\circ = H - \ell$, which implies that $\ell \cap B \subset \partial B$.

Conversely, suppose that $q \in \ell \cap B \subset \partial B$. Because B is a convex body, we can choose some $y \in B^\circ$, which cannot lie on ℓ by assumption. Let H denote the closed half-plane cut out by ℓ that contains y. For the sake of contradiction, suppose that there were some $x \in B - H$. Then x and y would lie strictly on opposite sides of ℓ, yielding a unique point $\{z\} = \ell \cap (x, y)$. By Lemma 2.3(c), we then have $z \in (x, y) \subset B^\circ$, which contradicts $\ell \cap B \subset \partial B$. Thus $B \subset H$, so ℓ supports B at q.

(c) Applying (a), it suffices to show that ℓ supports B. For the sake of contradiction, suppose that $y \in \ell \cap B^\circ$. Because $[u, v] \subset \partial B$, the point y must lie on $\ell - [u, v]$; without loss of generality, we assume that u lies between v and y. Then Lemma 2.3(c) gives $u \in (v, y) \subset B^\circ$, since $y \in B^\circ$. But this contradicts $u \in \partial B$, so we must have $\ell \cap B^\circ = \emptyset$ and thus $\ell \cap B \subset \partial B$. Then (b) implies that ℓ supports B at u.

(d) Let $\Pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. Define the quantity $m = \max \Pi(B)$ (this exists because $B \neq \emptyset$ is compact) and the line $\ell = \Pi^{-1}(m)$. Then $\ell \cap B$ is a nonempty subset of ∂B, so (b) shows that ℓ supports B. Thus (a) shows that $\ell \cap B$ is a face of B, which is of the form $\ell \cap B = [u, v]$ for some $u, v \in \ell$. Therefore, u is an extreme point of the face $\ell \cap B$, and thus an extreme point of B. ■

We now prove the bound $\varpi \geq 3$ by mimicking the classical straightedge-compass construction of an equilateral triangle. As with the upper bound, this is straightforward, but a more careful analysis is required to classify all $X \in \mathcal{M}$ satisfying $\varpi(X) = 3$.

Theorem 3.8. *For any norm $X \in \mathcal{M}$, we have $\varpi(X) \geq 3$. Moreover, we have $\varpi(X) = 3$ if and only if B_X is a linearly regular hexagon.*

Proof. By Lemma 3.7(d), we may choose any extreme point $u \in B_X$; then $u \in \partial B_X$, so we have $||u||_X = 1$. Note that $||0||_X = 0$, $||2u||_X = 2$, and $0, 2u \in \partial B_X + u$.

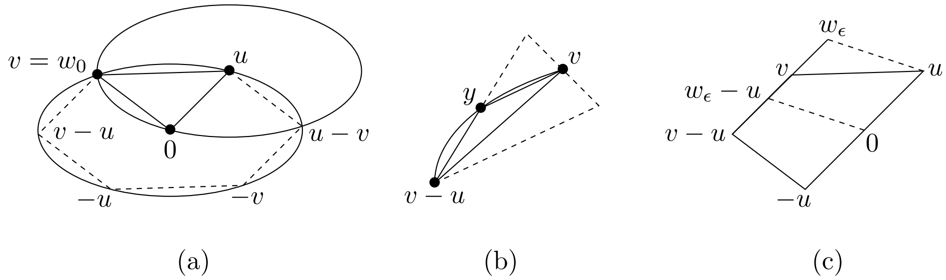

Figure 4. Three stages in the proof of Theorem 3.8.

Since ∂B_X is connected, there exists some vector $v \in \partial B_X + u$ with $||v||_X = 1$. Then $v - u \in \partial B_X$, so $||v-u||_X = 1$. This construction is illustrated in Figure 4(a). For any $0 \le \epsilon < 1$, let $w_\epsilon = v + \epsilon u$ and define the hexagon

$$H_\epsilon = \text{conv}\{u, w_\epsilon, v-u, -u, -v, u-v\}.$$

The vertices are listed in cyclic order (see Figure 4(a)), which we use to calculate

$$\begin{aligned}\text{len}_X(\partial H_\epsilon) &= ||w_\epsilon - u||_X + ||v-u-w_\epsilon||_X \\ &\qquad\qquad + ||-v||_X + ||u-v||_X + ||u||_X + ||v||_X \\ &= ||w_\epsilon - u||_X + (1+\epsilon)||u||_X + 4 = ||w_\epsilon - u||_X + 5 + \epsilon.\end{aligned}$$

If $w_\epsilon \in B_X$, then we have $H_\epsilon \subset B_X$ by the convexity of B_X, so Lemma 2.1 gives

$$\text{len}_X(\partial B_X) \ge \text{len}_X(\partial H_\epsilon) = ||w_\epsilon - u||_X + 5 + \epsilon. \tag{2}$$

Since $w_0 = v \in B_X$ and $||w_0 - u||_X = ||v-u||_X = 1$, this gives $\text{len}_X(\partial B_X) \ge 6$. This proves that $\varpi(X) \ge 3$. Now assume that $\varpi(X) = 3$, i.e., $\text{len}_X(\partial B_X) = 6$. Notice that $H_0 = \text{conv}\{u, v, v-u, -u, -v, u-v\}$ is a linearly regular hexagon, which is also X-equilateral, meaning that any adjacent vertices are X-distance 1 apart. Under the assumption that $\varpi(X) = 3$, we will prove that $B_X = H_0$.

Let φ denote the shorter path along ∂B_X between any two adjacent vertices of H_0. Then we have $\text{len}_X \varphi \ge 1$, since the endpoints of φ are X-distance 1 apart. However, $\text{len}_X(\partial B_X) = 6$ is the sum of the lengths of these six paths, so we must have equality $\text{len}_X \varphi = 1$ for each path. Now let φ be the shorter path along ∂B_X from v to $v-u$. Since $\text{len}_X \varphi = 1$, there is some $y \in \varphi$ with $d_X(v,y) = 1/2$. We also get

$$1 = d_X(v, v-u) \le d_X(v,y) + d_X(y, v-u) \le \text{len}_X \varphi = 1,$$

since $v, y, v-u$ is a partition of φ. We have equality throughout, so $d_X(v,y) = 1/2$ implies that $d_X(y, v-u) = 1/2$. Thus $2(v-y), 2(y-v+u) \in \partial B_X$. Notice that

$$u = \tfrac{1}{2} \cdot 2(v-y) + \tfrac{1}{2} \cdot 2(y-v+u) \in \big(2(v-y), 2(y-v+u)\big).^4$$

This process is illustrated in Figure 4(b). Because u is an extreme point, we must have $u = 2(v-y) = 2(y-v+u)$ and thus $y = v - u/2$, the midpoint of $[v, v-u]$. Hence $y \in \varphi \cap (v, v-u)$ and therefore $\varphi = [v, v-u]$ by Lemmas 2.3(e) and (b).

[4]For any vectors $p, q \in \mathbb{R}^2$, we have $p + q = u$ if and only if u is the midpoint of $[2p, 2q]$.

If we can prove that v is an extreme point, then we can iterate this whole process, i.e., show that $[v-u,-u] \subset \partial B_X$ and then show that $v-u$ is an extreme point, etc. In total, this shows that $\partial B_X = \partial H_0$ and thus $B_X = H_0$ is a linearly regular hexagon.

Consider the line $\ell = \{v + tu : t \in \mathbb{R}\}$ and note that $v, v-u \in \ell$. Since we have shown that $[v, v-u] = \varphi \subset \partial B_X$, Lemma 3.7(c) states that $\ell \cap B_X$ is a face of B_X and a closed line segment. Hence, to show that v is an extreme point of B_X, we need only show that v is an extreme point of $\ell \cap B_X$, i.e., an endpoint of this line segment. But if v is not an endpoint, then there exists $0 < \epsilon < 1$ with $w_\epsilon = v + \epsilon u \in \ell \cap B_X$. Then we have $w_\epsilon - u = \epsilon v + (1-\epsilon)(v-u) \in [v, v-u] \subset \partial B_X$, so (2) becomes

$$6 = \text{len}_X(\partial B_X) \geq ||w_\epsilon - u||_X + 5 + \epsilon = 6 + \epsilon > 6.$$

This is a contradiction, so v must be an endpoint of $\ell \cap B_X$ (see Figure 4(c)). ■

Finally, we have proven Gołąb's theorem: that $3 \leq \varpi(X) \leq 4$ for any $X \in \mathcal{M}$, with equality on the right if and only if B_X is a parallelogram (centered at the origin), and equality on the left if and only B_X is a linearly regular hexagon. In the next section, we depart from Gołąb's classical results—as retold by Schäffer—and focus instead on extending the result of [**3**, Proposition 3], where the constant π retakes center stage.

4. WHICH NORMS ARE EUCLIDEAN? Inner products are symmetric, bilinear functions $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that $v \mapsto \sqrt{\langle v, v\rangle}$ is a norm on $\mathbb{R}^n$. If a norm X arises from an inner product in this way, it is said to be *Euclidean*. The renowned "parallelogram law" states that a norm X is Euclidean if and only if, for all $u, v \in \mathbb{R}^n$,

$$2||u||_X^2 + 2||v||_X^2 = ||u+v||_X^2 + ||u-v||_X^2. \tag{3}$$

Since (3) only involves two vectors at a time, a norm on $\mathbb{R}^n$ is Euclidean if and only if its restriction to any two-dimensional subspace is Euclidean. This gives a special role to geometric conditions for norms on $\mathbb{R}^2$ that precisely classify the Euclidean norms. We will write $\mathcal{E} \subset \mathcal{M}$ to denote the set of Euclidean norms on $\mathbb{R}^2$.

The standard inner product on $\mathbb{R}^n$ is simply given by $\langle u, v\rangle = u_1v_1 + \cdots + u_nv_n$, and it induces the Euclidean ℓ^2 norm. Any inner product admits an orthonormal basis (using the Gram–Schmidt process), which uniquely characterizes this inner product. Mapping this basis to $e_1, \ldots, e_n$, we can see that any Euclidean norm on $\mathbb{R}^n$ is linearly equivalent to ℓ^2. But if $\langle \cdot, \cdot \rangle$ is an inner product and $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then the map $(u, v) \mapsto \langle T^{-1}u, T^{-1}v\rangle$ is also an inner product. Thus Euclidean norms are closed under linear equivalence, so a norm X on $\mathbb{R}^n$ is Euclidean if and only if X is linearly equivalent to the ℓ^2 norm (i.e., B_X is an ellipsoid). Thus for $n = 2$, we have

$$X \in \mathcal{E} \implies \varpi(X) = \varpi(\ell^2) = \pi,$$

by Lemma 3.3. But the converse does not hold, since taking $t = \pi - 3$ in Example 1.1 yields a norm X, where $\varpi(X) = \pi$ and B_X is a hexagon.

The fundamental notions carried by an inner product are distance and orthogonality, so it makes sense to ask if the relation of "orthogonality" can be generalized to pairs of vectors in any normed space. In fact, there are several ways to do so; a particularly useful one takes inspiration from the classic theorem of Euclidean geometry that states, "a line tangent to a circle is perpendicular to the coincident radius."

Definition 4.1. Let X be a norm on $\mathbb{R}^n$ and consider two nonzero vectors $x, y \in \mathbb{R}^n$. We will say that y is *Birkhoff*[5] *orthogonal* to x, written $y \vdash_X x$, if $||x + ty||_X \geq ||x||_X$

[5] Although it bears Birkhoff's name, this concept goes back at least to Carathéodory [**2**].

for all $t \in \mathbb{R}$. This is equivalent to saying that the line $\{x + ty : t \in \mathbb{R}\}$ is "tangent" to the ball of radius $||x||_X$, meaning that it intersects the boundary but not the interior.[6] While the relation $\vdash_X$ depends intimately on the norm X, we will suppress the subscript and just write $y \vdash x$ when there is no chance of confusion.

For a Euclidean norm, we have $y \vdash x$ if and only if $\langle x, y \rangle = 0$, so $\vdash$ is symmetric. It is important to note that this symmetry does not hold in general: we cannot freely interchange x and y. To remember the order in which to write them, notice that $y \vdash x$ is a pictograph of the line $\{x + ty : t \in \mathbb{R}\}$ meeting with the radius $\{tx : t \in [0, 1]\}$. We will return to the question of symmetry for $\vdash$ later.

Now, we will classify circles, using Birkhoff orthogonality and quarter-turns in $\mathbb{C}$. In this next lemma, the notation of complex numbers will help to simplify our notation. In particular, for any nonzero $v \in \mathbb{C}$, we will consider the perpendicular line

$$\ell_v = \{v + tiv : t \in \mathbb{R}\}.$$

For any norm $X \in \mathcal{M}$ and $v \in \partial B_X$, notice that $iv \vdash_X v$ if and only if $\ell_v \cap B_X^\circ = \emptyset$. This notation of ℓ_v will be useful in Lemmas 4.2 and 4.3, as well as in Theorem 4.4.

Lemma 4.2. *Suppose $X \in \mathcal{M}$ satisfies $iv \vdash_X v$ for all $v \in \partial B_X$. Then X is a positive multiple of ℓ^2. In particular, this implies that X is Euclidean.*

Proof. For brevity, we will omit some details. For any $u \in \mathbb{C}$ and $t \in [0, 1]$, let $D_t(u)$ be the (Euclidean) disk of diameter $[-tu, u]$. The reader may verify that $v \in D_0(u)^\circ$ if and only if ℓ_v intersects $(0, u)$. (Use the inscribed angle theorem or algebraically manipulate the inner product.) For any $u \in B_X$, Lemma 2.3(c) gives $(0, u) \subset B_X^\circ$, because we always have $0 \in B_X^\circ$.

For the sake of contradiction, suppose we have $u \in B_X$ and $v \in D_0(u)^\circ - B_X^\circ$. Then there exists some $w \in \ell_v \cap (0, u)$. Since $||v||_X \geq 1$, we can define

$$\widehat{v} = \frac{v}{||v||_X} \in \partial B_X \qquad \text{and} \qquad \widehat{w} = \frac{w}{||v||_X} \in (0, u).$$

Then $\widehat{w} \in \ell_{\widehat{v}} \cap (0, u) \subset \ell_{\widehat{v}} \cap B_X^\circ$, which contradicts $i\widehat{v} \vdash \widehat{v}$. Therefore, if $u \in B_X$, then $D_0(u)^\circ \subset B_X^\circ$ and hence $D_0(u) \subset B_X$, since B_X is closed. Now, we may define

$$m = \sup\{t \in [0, 1] : u \in B_X \implies D_t(u) \subset B_X\}. \tag{4}$$

Then $m \in [0, 1]$ and since B_X is closed, we see that $u \in B_X$ implies $D_m(u) \subset B_X$. Since B_X is compact, there exists $w \in B_X$ with maximal ℓ_2-norm. If $m = 1$, then

$$B_X \subset |w| \cdot B_{\ell^2} = D_1(v) \subset B_X.$$

Thus $B_X = |w| \cdot B_{\ell^2}$ and hence X is a positive multiple of ℓ^2 (assuming that $m = 1$).

To complete the proof, we will assume that $m < 1$ and derive a contradiction. Let

$$a = \frac{1 + m}{1 - m} \qquad \text{and} \qquad s = \left(\frac{a}{a + 1}\right)^2 = \left(\frac{m + 1}{2}\right)^2.$$

The reader may check that $m < s < 1$. For an arbitrary point $u \in B_X$, we will prove that $D_s(u) \subset B_X$, which contradicts the definition of m in (4). We assume that $u \neq 0$,

[6] When $n = 2$, Lemma 3.7(b) shows that this notion of "tangency" is precisely that of a supporting line.

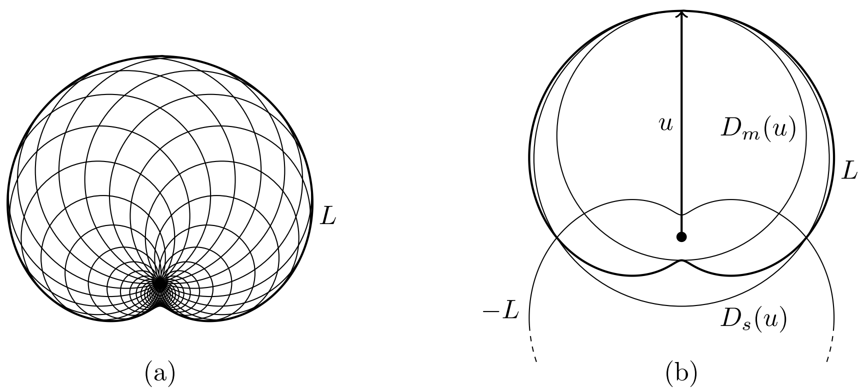

Figure 5. Sweeping out a limaçon to get a larger disk.

since $D_s(0) = \{0\} \subset B_X$ is clear. We will now use polar coordinates relative to u, meaning that the coordinates (r, θ) denote the point $re^{i\theta}u$. In this notation, we define

$$\pm L = \left\{ (r, \theta) \in \mathbb{R}^2 : r \leq \frac{a \pm \cos\theta}{a+1} \right\}. \tag{5}$$

The $\pm$ on each side makes sense, since negating $\cos\theta$ corresponds to a rotation by π, which is equivalent to reflecting L through the origin to get $-L$. The boundary curve ∂L is a limaçon, which possesses the lovely property that it occurs as the "envelope" of the family of curves $\{\partial D_0(v) : v \in \partial D_m(u)\}$ [**8**]. This is illustrated in Figure 5(a). Our use of this fact will be confined to the following statement:

$$L = \bigcup_v D_0(v), \text{ where the union ranges over all } v \in \partial D_m(u).$$

Since $v \in \partial D_m(u) \subset B_X$ implies $D_0(v) \subset B_X$, we see that $L \subset B_X$. Because B_X is symmetric about the origin, we also have $-L \subset B_X$. Next, we consider the function

$$f(t) = \frac{(1-s)t + \sqrt{4s + (1-s)^2 t^2}}{2}.$$

This function is cooked up in order to describe the disk $D_s(u)$ in our polar coordinates:

$$D_s(u) = \left\{ (r, \theta) \in \mathbb{R}^2 : r \leq f(\cos\theta) \right\}. \tag{6}$$

To see why this holds, the reader should use the law of cosines to show that the curve $\partial D_s(u)$ is described by the equation $r^2 = (1-s)r\cos\theta + s$, then use the quadratic formula to find the value of $r > 0$, yielding the equation $r = f(\cos\theta)$ for $\partial D_s(u)$. The reader should also verify that f is convex and use this to prove the inequalities

$$t \in [0,1] \Longrightarrow f(t) \leq \frac{a+t}{a+1} \quad \text{and} \quad t \in [-1,0] \Longrightarrow f(t) \leq \frac{a-t}{a+1}. \tag{7}$$

Considering the sign of $\cos\theta$, we may now compare (5) and (6) via (7), to see that

$$\text{if } \theta \in \left[\tfrac{-\pi}{2}, \tfrac{\pi}{2}\right], \text{ then } (r,\theta) \in D_s(u) \Longrightarrow (r,\theta) \in L \subset B_X;$$
$$\text{if } \theta \in \left[\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right], \text{ then } (r,\theta) \in D_s(u) \Longrightarrow (r,\theta) \in -L \subset B_X.$$

Therefore $D_s(u) \subset B_X$, as desired. This argument is illustrated in Figure 5(b). ■

Quarter-turn symmetry. We now return to ϖ, to expand upon [**3**, Proposition 3], which states that $\varpi(X) \geq \pi$ whenever $iX = X$ (this means that pushing forward by a quarter-turn does not change lengths). Recall that $iX = X$ if and only if $iB_X = B_X$ (i.e., B_X has quarter-turn symmetry). We begin with a crucial lemma, which compares lengths under such a norm to the measures of Euclidean angles.

Lemma 4.3. *Fix any norm $X \in \mathcal{M}$ with $iX = X$.*

(a) *Suppose that $v \neq 0$ and $p, q \in \ell_v$ are distinct. If θ denotes the angle measure between the vectors p and q, then we have $d_X(p, q) > \theta \cdot ||v||_X$.*

(b) *If φ is a path along ∂B_X that sweeps out an angle of θ (centered at the origin), then* $\mathrm{len}_X \varphi \geq \theta$. *If φ is a polygonal path (and not just a point), then* $\mathrm{len}_X \varphi > \theta$.

Proof. The main argument in this proof follows that of [**3**, Proposition 3] very closely.

(a) We first consider the Euclidean geometry. Define $r = |p|$, $s = |q|$, $b = |p - q|$ and $h = |v|$, as shown in Figure 6(a). We have $bh = rs \sin \theta$, because both sides equal twice the area of $\triangle 0pq$. By the law of cosines in the same triangle, we also have $b^2 = r^2 + s^2 - 2rs \cos \theta$. Notice that $t + \frac{1}{t} \geq 2$ for all $t > 0$. Therefore, we have

$$\frac{b}{h} = \frac{b^2}{bh} = \frac{r^2 + s^2 - 2rs \cos \theta}{rs \sin \theta} = \frac{\frac{r}{s} + \frac{s}{r} - 2 \cos \theta}{\sin \theta} \geq \frac{2 - 2 \cos \theta}{\sin \theta}.$$

Since p and q are not parallel, we have $\theta < \pi$. The tangent half-angle formula gives

$$\frac{b}{h} \geq \frac{2 - 2 \cos \theta}{\sin \theta} = 2 \tan(\theta/2) > \theta,$$

where the last inequality follows because $\tan t > t$ for all $0 < t < \pi/2$.

Now, we must consider length in terms of X. Since $p, q \in \ell_v$, we have $p - q = tiv$ for some $t \in \mathbb{R}$. For any norm $Y \in \mathcal{M}$ with $iY = Y$ (e.g., X or ℓ^2), it follows that

$$||p - q||_Y = ||tiv||_Y = |t| \cdot ||v||_Y.$$

This allows us to translate the result for ℓ^2 to any norm X with $iX = X$:

$$||p - q||_X = |t| \cdot ||v||_X = \frac{|p - q|}{|v|} \cdot ||v||_X = \frac{b}{h} \cdot ||v||_X > \theta \cdot ||v||_X.$$

(b) First suppose that $\varphi = [p, q]$ with $p \neq q$. Let ℓ denote the line through p and q. Because $[p, q] \subset \partial B_X$, Lemma 3.7(c) states that $\ell \cap B_X$ is a face of B_X. Moreover, since $\ell \cap B_X$ is a proper face of B_X, we have $\ell \cap B_X \subset \partial B_X$ and thus $\ell \cap B_X^\circ = \emptyset$. In particular, this shows that $0 \notin \ell$, so we can write $\ell = \ell_v$, where v is the unique point along ℓ of minimal ℓ^2-norm. Since $v \notin B_X^\circ$, we have $||v||_X \geq 1$. Therefore (a) gives

$$\mathrm{len}_X \varphi = d_X(p, q) > \theta \cdot ||v||_X \geq \theta.$$

This proves the desired result whenever φ is a line segment. Since lengths are additive under concatenation of paths, we also get the desired result for any polygonal path.

In proving the general case, we may assume that $\theta < \pi$ (by the additivity of length). Then if φ goes from p to q, the angle θ swept out by φ is the angle between the vectors p and q (which is unchanged if p or q is scaled by a positive number). Fix some $\epsilon > 1$. We wish to find some norm $Y \in \mathcal{M}$, such that the unit ball B_Y is a convex polygon,

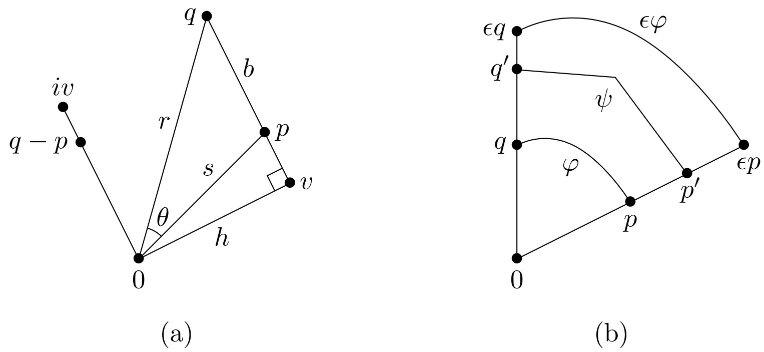

Figure 6. Lengths in terms of a norm X such that $iX = X$.

$iY = Y$, and $B_X \subset B_Y \subset \epsilon B_X$. Let $x_0, x_1, x_2, \ldots \in \partial(\epsilon B_X)$ be a dense sequence. Then $B_X \subset \text{conv}\{x_n : n \in \mathbb{N}\}^\circ$ and thus the sets $U_n = \text{conv}\{x_0, x_1, \ldots, x_n\}^\circ$ form an open cover of B_X indexed by $n \in \mathbb{N}$. Since $U_1 \subset U_2 \subset \cdots$ and B_X is compact, there exists some $n \in \mathbb{N}$ with $B_X \subset U_n$. Then we define Y by its unit ball

$$B_Y = \text{conv}\{ax_k : k = 0, 1, \ldots, n \text{ and } a = \pm 1 \text{ or } \pm i\}.$$

This is clearly a convex body with quarter-turn symmetry and $B_X \subset B_Y \subset \epsilon B_X$.

There are unique $p' \in \partial B_Y \cap [p, \epsilon p]$ and $q' \in \partial B_Y \cap [q, \epsilon q]$, which we can write as $p' = sp$ and $q' = tq$, for some $s, t \in [1, \epsilon]$. Let ψ denote the shorter path along ∂B_Y from p' to q'. Notice that ψ also sweeps out an angle of θ and B_Y is polygonal, so the above case gives $\text{len}_Y \psi > \theta$. Then $\psi \prec [p', \epsilon p] \bullet \epsilon\varphi \bullet [\epsilon q, q']$, where the latter is a convex path because it is a part of the boundary of $\{tv : t \in [0, \epsilon] \text{ and } v \in \varphi\}$. These convex paths are illustrated in Figure 6(b). Since $p' \in [p, \epsilon p]$, we have

$$d_Y(p', \epsilon p) \le d_Y(p, \epsilon p) = (\epsilon - 1)||p||_Y.$$

We analogously have $d_Y(q', \epsilon q) \le (\epsilon - 1)||q||_Y$. By Lemma 2.3(a), we then have

$$\begin{aligned}\theta < \text{len}_Y \psi &\le \text{len}_Y\big([p', \epsilon p] \bullet \epsilon\varphi \bullet [\epsilon q, q']\big)\\ &= d_Y(p', \epsilon p) + d_Y(q', \epsilon q) + \text{len}_Y(\epsilon\varphi)\\ &\le (\epsilon - 1)\big(||p||_Y + ||q||_Y\big) + \text{len}_Y(\epsilon\varphi).\end{aligned}$$

Since $B_X \subset B_Y$, we have $\text{len}_X \ge \text{len}_Y$ and $||\cdot||_X \ge ||\cdot||_Y$. Therefore,

$$\begin{aligned}\epsilon \cdot \text{len}_X \varphi \ge \epsilon \cdot \text{len}_Y \varphi &= \text{len}_Y(\epsilon\varphi)\\ &> \theta - (\epsilon - 1)\big(||p||_Y + ||q||_Y\big)\\ &\ge \theta - (\epsilon - 1)\big(||p||_X + ||q||_X\big).\end{aligned}$$

Taking the limit as $\epsilon \to 1$ yields the desired inequality $\text{len}_X \varphi \ge \theta$. ■

This lemma carries most of the burden of proving that $\varpi(X) \ge \pi$ when $iX = X$, as well as classification of the equality case. But the condition $iX = X$ is not ideal, in that it is not preserved under linear equivalence (consider ellipses). Thus we define

$$\mathcal{Q} = \{Y \in \mathcal{M} : Y \text{ is linearly equivalent to some } X \in \mathcal{M} \text{ with } iX = X\}.$$

Then $\mathcal{Q}$ is obviously closed under linear equivalence, so it provides a "coordinate-free" notion of norms with quarter-turn symmetry.[7] With this notion in hand, the quarter-turn symmetry results can be extended to $\mathcal{Q}$, giving a characterization of $\mathcal{E}$ in terms of ϖ.

It should be noted that the inequality in Theorem 4.4(a) is from [**3**, Proposition 3]. However, the equality case is of my own invention, at least by this method of proof.[8]

Theorem 4.4. (a) *For any norm $X \in \mathcal{M}$ such that $iX = X$, we have $\varpi(X) \geq \pi$. Moreover, we have $\varpi(X) = \pi$ if and only if X is a positive multiple of ℓ^2.*

(b) *We have $\varpi(\mathcal{Q}) = [\pi, 4]$ and $\mathcal{E} = \{X \in \mathcal{Q} : \varpi(X) = \pi\}$.*

Proof. (a) Note that $\varpi(X) \geq \pi$ follows at once from Lemma 4.3(b) with $\varphi = \partial B_X$, which sweeps out an angle of $\theta = 2\pi$. If X is a positive multiple of ℓ^2, then $X \in \mathcal{E}$ and thus $\varpi(X) = \pi$. Conversely, now suppose that X is not a positive multiple of ℓ^2. By the contrapositive of Lemma 4.2, there exists some $v \in \partial B_X$ with $\ell_v \cap B_X^\circ \neq \emptyset$. Because the line ℓ_v intersects the interior of B_X, it must intersect the boundary ∂B_X in at least two points, so there is some $u \in \ell_v \cap \partial B_X$ with $u \neq v$. Let φ and ψ denote the shorter and longer paths along ∂B_X from u to v, respectively. If θ is the angle formed by u and v, then ψ sweeps out an angle of $2\pi - \theta$. It follows that

$$\mathrm{len}_X(\partial B_X) = \mathrm{len}_X \psi + \mathrm{len}_X \varphi \geq 2\pi - \theta + d_X(u, v) > 2\pi - \theta + \theta = 2\pi$$

by Lemmas 4.3(b) and (a). Therefore, we get the *strict* inequality $\varpi(X) > \pi$.

(b) If $Y \in \mathcal{Q}$, then Y is linearly equivalent to some $X \in \mathcal{M}$ such that $iX = X$, so $\varpi(Y) = \varpi(X) \geq \pi$ by Lemma 3.3 and (a). Since the upper bound of $\varpi(Y) \leq 4$ was already established in Theorem 3.5, we have $\varpi(\mathcal{Q}) \subset [\pi, 4]$. To show the reverse, notice that ℓ^p has quarter-turn symmetry for all $p \in [1, \infty]$. We know that $\varpi(\ell^2) = \pi$ and $\varpi(\ell^1) = 4$, so the intermediate value theorem implies that

$$[\pi, 4] \subset \varpi\big(\{\ell^p : 1 \leq p \leq 2\}\big) \subset \varpi(\mathcal{Q}).$$

This uses the fact that $p \mapsto \ell^p \mapsto \varpi(\ell^p)$ is a composition of continuous functions, where $\mathcal{M}$ is given the structure of a metric space by measuring the Hausdorff distance between unit balls. The continuity of $p \mapsto \varpi(\ell^p)$ can also be directly observed from an integral formula found in [**1**]. Either way, we omit the details.

We already know that $\ell^2 \in \mathcal{Q}$ and $\varpi(\ell^2) = \pi$. But since $\{X \in \mathcal{Q} : \varpi(X) = \pi\}$ is closed under linear equivalence, this implies that $\mathcal{E} \subset \{X \in \mathcal{Q} : \varpi(X) = \pi\}$. Conversely, consider $Y \in \mathcal{Q}$ with $\varpi(Y) = \pi$. Then Y is linearly equivalent to some $Z \in \mathcal{M}$ with $iZ = Z$. We have $\varpi(Z) = \varpi(Y) = \pi$ by Lemma 3.3, so Z is a positive multiple of ℓ^2 by (a). Hence, Z is Euclidean and therefore Y is Euclidean as well. ■

Radon norms. In this final section, we will recount one more result without proof, providing an interesting parallel to Theorem 4.4. Recall that the relation of Birkhoff orthogonality is not generally symmetric. In fact, if X is a norm on $\mathbb{R}^n$ with $n \geq 3$, then $\vdash_X$ is symmetric if and only if X is Euclidean [**12**, Theorem 3.4.10]. The situation differs in two dimensions, where there are many such norms that are non-Euclidean:

Definition 4.5. A norm $X \in \mathcal{M}$ is said to be *Radon* if the relation $\vdash_X$ is symmetric. We will write $\mathcal{R}$ to denote the set of all Radon norms on $\mathbb{R}^2$.

[7] A more coordinate-free definition is $X \in \mathcal{Q}$ if and only if $SX = X$ for some $S \in \mathrm{GL}(2, \mathbb{R})$ of order 4.

[8] The inequality in Theorem 4.4(a) is also proved in [**6**, Theorem 5], in part, using an inequality from [**5**]. While it is not mentioned in [**6**], the main result of [**5**] also includes an equality case, which can be combined with [**6**, Theorem 5] to yield an alternative proof of the equality case in Theorem 4.4(a).

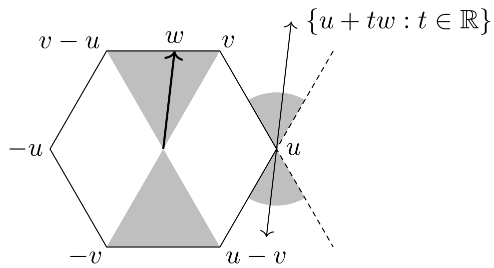

Figure 7. A linearly regular hexagon is the unit ball of a Radon norm.

Going back to the definition of Birkhoff orthogonality, we can see that $\mathcal{R}$ is closed under linear equivalence and $\mathcal{E} \subset \mathcal{R}$. To see that Definition 4.5 is not just an alternative characterization of Euclidean norms, we now return to one of our central examples:

Example 4.6. Consider a norm whose unit ball is the linearly regular hexagon

$$H = \text{conv}\{u, v, v - u, -u, -v, u - v\},$$

where $u, v \in \mathbb{R}^2$ are linearly independent. To show that this norm is Radon, it suffices to prove that $\vdash$ is a symmetric relation for vectors in the boundary ∂H. There are two cases to address, in order to understand the relation of orthogonality in this norm:

- A point w lying in the interior of an edge clearly admits only one supporting line. Translating this line to the origin, we find exactly two points $x \in \partial H$ with $x \vdash w$, both of which are vertices of H. For example, if $w \in (v, v - u)$, then $x = \pm u$.
- A vertex x admits many supporting lines. These lines are precisely those that sweep out the "double wedge" region formed by extending the edges out from x, as shown in Figure 7. If we translate x to the origin, this "double wedge" then intersects ∂H precisely in the edges that don't contain $\pm x$. For example, if $x = u$ and $w \in \partial H$, then $w \vdash x$ if and only if $\pm w \in [v, v - u]$.

Combining these two cases, we can observe that $\vdash$ is indeed a symmetric relation.

This example shows that Radon norms can have $\varpi = 3$ or π, so it seems reasonable to expect that they would assume all pi-values between these two. In fact, this interval of $[3, \pi]$ comprises all of the possible pi-values that are assumed by Radon norms:

Theorem 4.7. *We have* $\varpi(\mathcal{R}) = [3, \pi]$ *and* $\mathcal{E} = \{X \in \mathcal{R} : \varpi(X) = \pi\}$.[9]

The reader should note a striking similarity between this result and Theorem 4.4(b). Since norms with quarter-turn symmetry have been studied far less than Radon norms, this resemblance raises the question of whether other results about $\mathcal{R}$ (for examples, see [**2**] or [**9**]) have some sort of analogue concerning $\mathcal{Q}$. I would be thrilled if this article prompts any such exploration.

To situate this relationship, a certain viewpoint may be useful. We have frequently written $iX = X$ to denote the condition of quarter-turn symmetry, and more generally, a norm is in $\mathcal{Q}$ if and only if it is "compatible" with some complex structure on $\mathbb{R}^2$. Martini and Swanepoel have described an elegant framework for Radon norms in [**9**],

[9] The accessible survey [**2**] gives a lovely account of some aspects of the theory that we have totally ignored, and contains the original references for this result, which is quoted therein as Corollary 4. This result is also discussed in the more comprehensive survey [**10**], where it is Proposition 48.

which essentially says that a norm is in $\mathcal{R}$ if and only if it is "compatible" with some symplectic structure on $\mathbb{R}^2$. We have utterly ignored notions of duality in this article, but a norm and its dual are often viewed in the same plane by way of an inner product. Considering the trinity of complex, symplectic, and Euclidean/Riemannian geometry, it would not be surprising for these structures to exhibit some interplay. We can already see one such result as a straightfoward corollary of the related theorems on $\mathcal{Q}$ and $\mathcal{R}$:

Corollary 4.8. *If $X \in \mathcal{M}$ is Radon and has quarter-turn symmetry (in some basis), then X is Euclidean. Stated more compactly, we have $\mathcal{Q} \cap \mathcal{R} = \mathcal{E}$.*

Proof. Theorems 4.4 and 4.7 give $\pi \leq \varpi(X) \leq \pi$, so $\varpi(X) = \pi$. Then the equality case of either theorem implies that X is Euclidean. ■

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REFERENCES

1. Adler, C. L., Tanton, J. (2000). π is the minimum value for pi. *Coll. Math. J.* 31(2): 102–106.
2. Álvarez Paiva, J. C., Thompson, A. (2005). On the perimeter and area of the unit disc. *Amer. Math. Monthly.* 112(2): 141–154.
3. Duncan, J., Luecking, D., McGregor, C. On the values of pi for norms on $\mathbb{R}^2$. *Coll. Math. J.* 35(2): 84–92.
4. Falconer, K. J. (1985). *The Geometry of Fractal Sets*. Cambridge Tracts in Mathematics, 85. Cambridge: Cambridge Univ. Press.
5. Firey, W. J. (1963). The mixed area of a convex body and its polar reciprocal. *Israel J. Math.* 1(3): 201–202.
6. Ghandehari, M., Martini, H. (2019). On self-circumferences in Minkowski planes. *Extracta Math.* 34(1): 19–28.
7. Gołąb, S. (1932). Zagadnienia metryczne geometrji Minkowskiego. *Prac. Akad. Gór. w Krak.* 6: 1–79.
8. Lockwood, E. H. (1961). *A Book of Curves.* Cambridge: Cambridge Univ. Press.
9. Martini, H., Swanepoel, K. J. (2006). Antinorms and Radon curves. *Aequationes Math.* 72(1-2): 110–138.
10. Martini, H., Swanepoel, K. J., Weiß, G. (2001). The geometry of Minkowski spaces—a survey. I. *Expo. Math.* 19(2): 97–142.
11. Schäffer, J. J. (1967). Inner diameter, perimeter, and girth of spheres. *Math. Ann.* 173(1): 59–79.
12. Thompson, A. C. (1996). *Minkowski Geometry.* Encyclopedia of Mathematics and Its Applications, 63. Cambridge: Cambridge Univ. Press.

NIKHIL HENRY BUKOWSKI SAHOO hails from the East Bay, where he attended the Peralta Community Colleges and where he eventually earned a B.A. in mathematics from the University of California, Berkeley. He is currently a Ph.D. student in mathematics at Cornell University studying equivariant symplectic topology.
Cornell University, Ithaca NY 14850
nhs58@cornell.edu