# A Whole Lot of Values for Pi

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## Backdrop

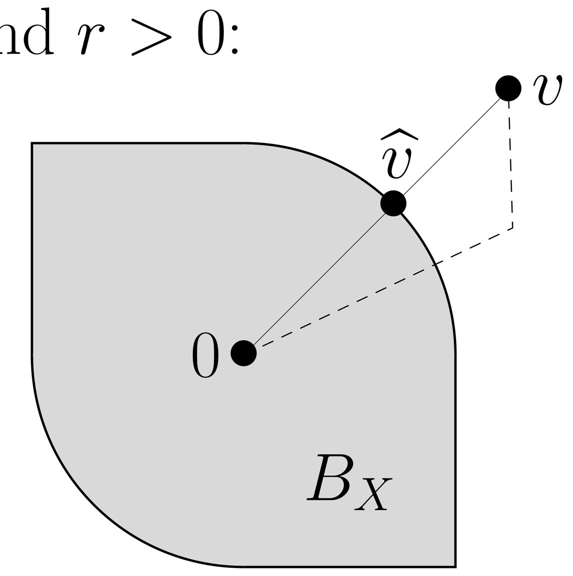
This work was conducted independently and hence funded mainly by my own privilege. I did this work on stolen Ohlone land—specifically belonging to the Chochenyo people—during a fatal pandemic, as the movement for Black lives rose up across the world.

I find it important to acknowledge the exceptional position that allowed me the free time for this work, while many others were out of work and fighting for their lives. The idea that "good" mathematicians have found more time to work during the pandemic has gone around: it must be confronted and rejected, just as we must always confront privilege and reject white supremacy, as often as they might occur.

## What's in a norm?

A norm $X$ on $\mathbb{R}^2$ is a function $||\cdot||_X : \mathbb{R}^2 \to [0, \infty)$, such that for any $u, v \in \mathbb{R}^2$ and $r > 0$:

- $||ru||_X = r||u||_X$;
- $||u + v||_X \leq ||u||_X + ||v||_X$;
- $||v||_X = 0 \iff v = 0$.




It can also be described as a "metric compatible with vector space structure." A norm defines a unit ball

$$B_X = \{v \in \mathbb{R}^2 : ||v||_X \leq 1\}.$$

This set $B_X$ is a "centrally symmetric convex body," whose boundary is the unit circle

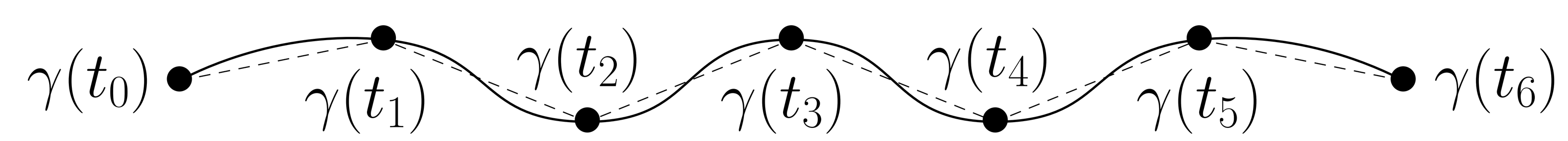
$$\partial B_X = \{v \in \mathbb{R}^2 : ||v||_X = 1\}.$$

Conversely, any such body $B$ defines a norm $X$: Let $||v||_X \in (0, \infty)$ (for any $v \neq 0$) be the unique positive real number such that $\hat{v} = v/||v||_X \in \partial B$.

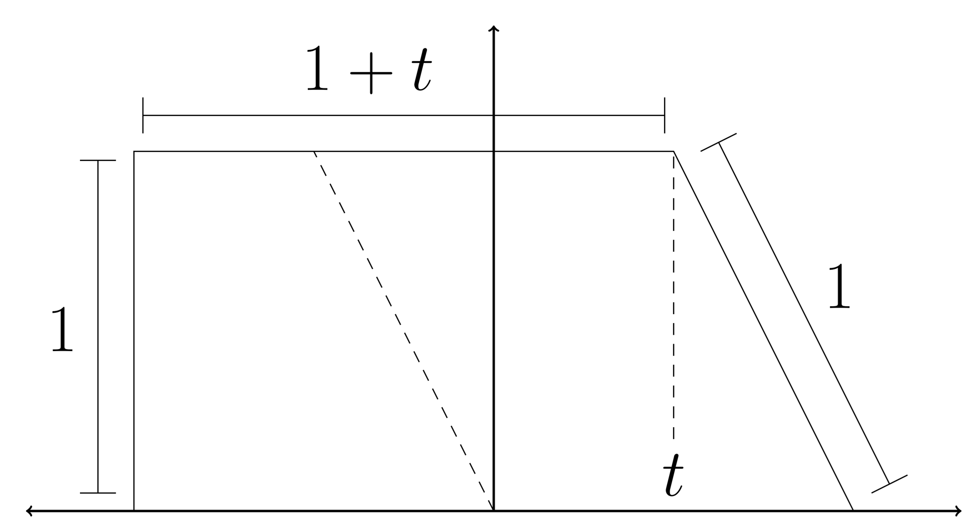
## Using a norm to measure … itself?

Given a norm $X$, we can define the length of a curve $\gamma : [0, 1] \to \mathbb{R}^2$ as

$$\text{len}_X(\gamma) = \sup \left\{ \sum_{k=1}^{n} \big|\big|\gamma(t_k) - \gamma(t_{k-1})\big|\big| : 0 = t_0 \leq t_1 \leq \ldots \leq t_n = 1 \right\}$$




If $\gamma_1$ and $\gamma_2$ are convex curves with the same endpoints, where $\gamma_1$ lies in the convex hull of $\gamma_2$, we can show that $\text{len}_X(\gamma_1) \leq \text{len}_X(\gamma_2) < \infty$. We get a notion of "pi = circumference ÷ diameter" for any norm $X$, given by $\varpi(X) = \text{len}_X(\partial B_X)/2$. For $0 \leq t \leq 1$, consider an example:




$$B_X = \text{hull}\big(e_1, e_2 - e_1, te_1 + e_2, -e_1, e_1 - e_2, -te_1 - e_2\big)$$

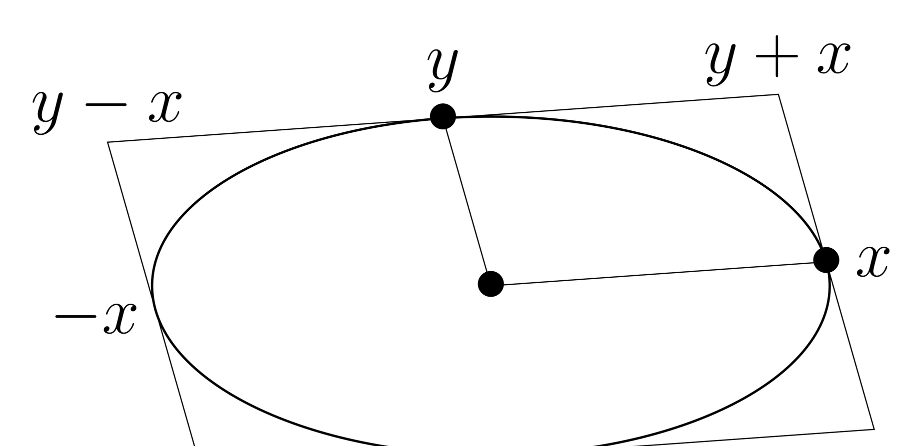
$$\varpi(X) = 1 + 1 + 1 + t = 3 + t$$

## Two constraining constructions

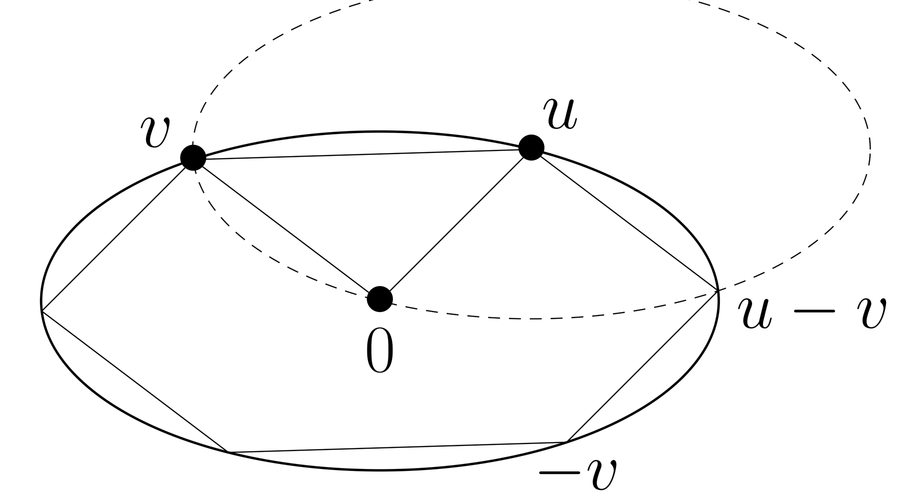
In 1932, the Polish geometer Stanisław Gołąb showed:

As $X$ ranges over all norms on $\mathbb{R}^2$, the values $\varpi(X)$ range over $[3, 4]$.

We saw above that every value in $[3, 4]$ is possible. To show that these are *all* of the possible values, we make use of an inscribed hexagon and a circumscribed parallelogram.




$\varpi(X) \leq 4$




$\varpi(X) \geq 3$

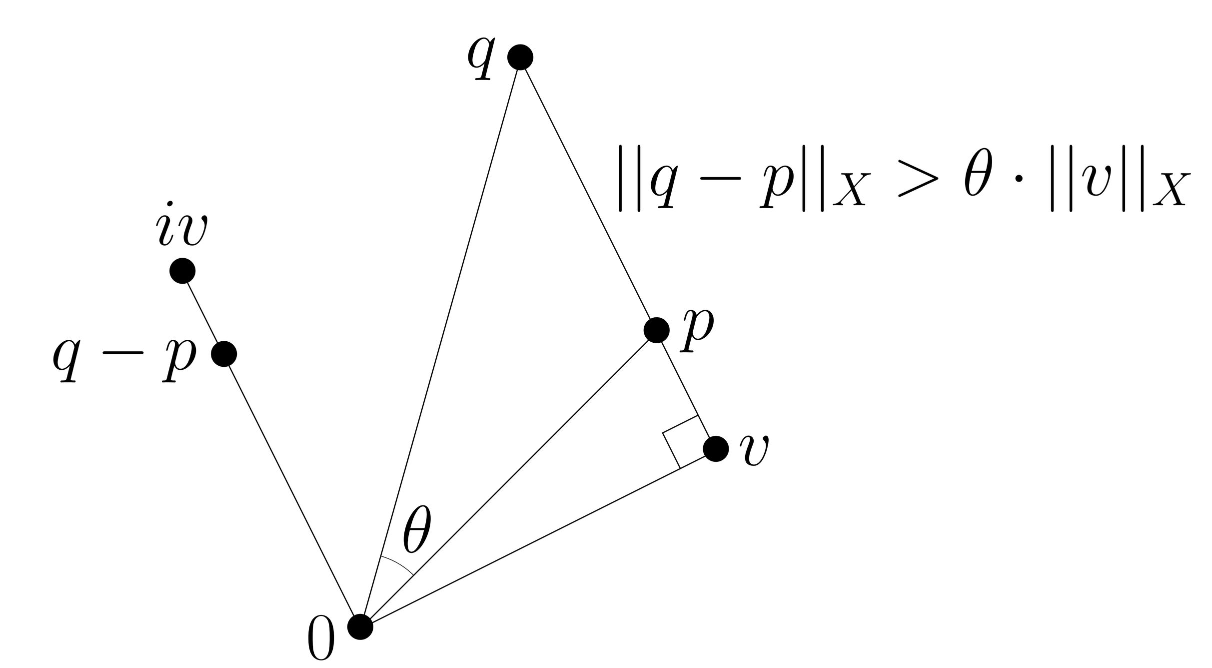
Gołąb also showed that equality occurs (on either side) *if and only if* $\partial B_X$ is precisely the inscribed or circumscribed polygon. Remarkably, this shows that the extremal cases are unique up to linear isomorphism.

## Quarter-turn symmetry

In [1], the authors consider a norm $X$ on $\mathbb{R}^2$ with quarter-turn symmetry. They show that:

- If $\partial B_X$ is polygonal and $\gamma$ is a portion of $\partial B_X$ subtending an angle $\theta$, then we have $\text{len}_X(\gamma) > \theta$.

Approximating $\partial B_X$ by polygons with quarter-turn symmetry then gives $\varpi(X) \geq \pi$ (the classic value).




Inspired by Gołąb's extremal results, I add that:

- Only when $B_X$ is a disk can we have this property from Euclidean geometry: given any $v \in \partial B_X$, the line $\ell_v$ through and perpendicular to $v$ doesn't intersect the interior of $B_X$ (a sort of tangency).

If there is any $v \in \partial B_X$ where $\ell_v$ is not "tangent," then a strict inequality applies, giving $\varpi(X) > \pi$. The (coordinate-free) contrapositive result is:

A norm $X$ on $\mathbb{R}^2$ is Euclidean *if and only if* $\varpi(X) = \pi$ and $X$ is invariant under an order-4 transformation of the plane ("a quarter-turn").

## References


[1] Duncan, J., Luecking, D., McGregor, C. On the Values of Pi for Norms on $\mathbb{R}^2$. *College Mathematics Journal.* 35(2): 84-92.

[2] Gołąb, Stanisław (1932). Zagadnienia metryczne geometrji Minkowskiego. *Prace Akademii Górniczej w Krakowie.* 6: 1–79.

[3] Thompson, A. C. (1996). *Minkowski Geometry. Encyclopedia of Mathematics and its Applications.* Cambridge University Press.