Smooth Resolution of Gelfand-Zetlin Polytopes and Toric Varieties

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Hilbert's 15$^{\text{th}}$ Problem

In the later 1800's, mathematician Hermann Schubert posed a problem:

Given four generic lines in 3D space, how many other lines intersect them?

To give an answer of 2, he used the non-rigorous *principle of conservation of number*, which states that the number of solutions (usually) does not depend on the particular initial conditions. In his famous 1900 address, David Hilbert asked that this solution be given a rigorous basis. This came about through study of spaces called **Grassmannians**, such as the space $Gr(2,4)$ of lines in 3D space. Given $\ell \in Gr(2,4)$, we define the subspace $X_\ell \subset Gr(2,4)$ of lines intersecting ℓ. Schubert's problem considers generic $\ell_1, \ell_2, \ell_3, \ell_4 \in Gr(2,4)$ and asks that we show $\#(X_{\ell_1} \cap X_{\ell_2} \cap X_{\ell_3} \cap X_{\ell_4}) = 2$.

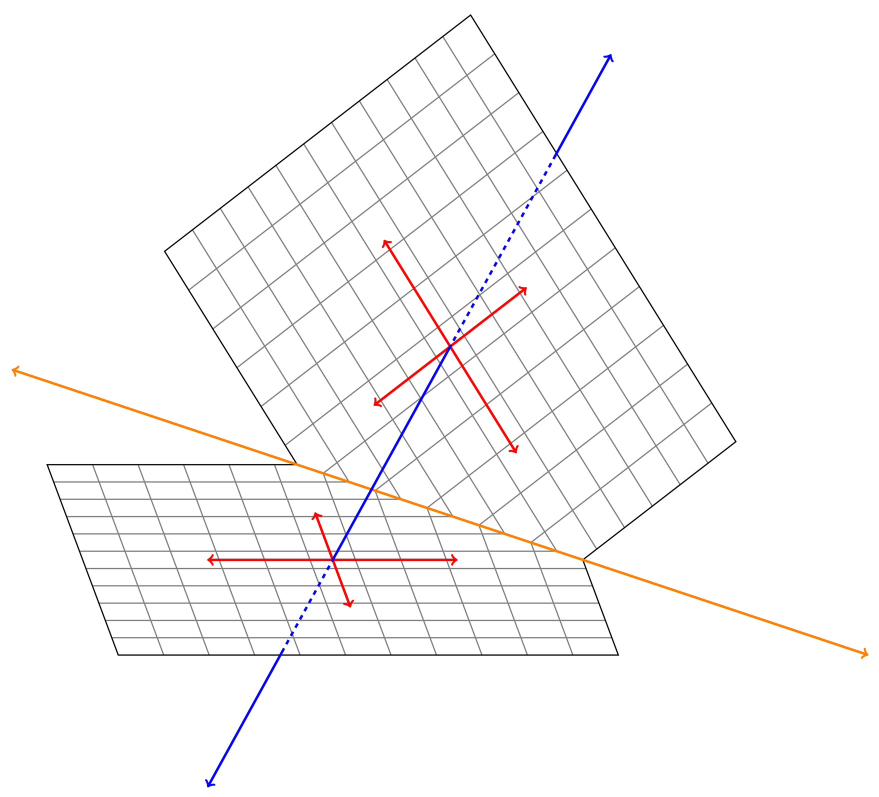

Fig. 1: The two solutions for Schubert's special arrangement of four lines.

The modern approach solves this counting problem via an algebraic object, called the cohomology ring $H^*(X)$ of the space X. In particular, we study certain elements of $H^*(Gr(k,n))$, called **Schubert classes**. These classes form an additive basis for the ring, indexed by diagrams of boxes that fit inside a $k \times (n-k)$ grid (X_ℓ corresponds to a single box $\square$). The counting problem then becomes to find the coefficients $c^\sigma_{\mu\lambda}$ in $[X_\mu] \cdot [X_\lambda] = \sum c^\sigma_{\mu\lambda}[X_\sigma]$.

$$X_\square \cdot X_\square \cdot X_\square \cdot X_\square = \left(X_{\square\square} + X_{\substack{\square\\\square}}\right) \cdot X_\square \cdot X_\square$$

$$= \left(X_{\substack{\square\square\\\square}} + X_{\substack{\square\square\\\square}}\right) \cdot X_\square = 2 \cdot X_{\substack{\square\square\\\square\square}}$$

Gelfand-Zetlin Polytopes

For reals $\lambda_1 \geq \ldots \geq \lambda_n$, the **Gelfand-Zetlin polytope** $GZ_n(\lambda_1, \ldots, \lambda_n)$ lies in a space of dimension $n(n-1)/2$, with coordinates $\lambda_i^{(j)}$ for $1 \leq i \leq j$ and $1 \leq j \leq n-1$. The Gelfand-Zetlin polytope is cut out by the inequalities

$$\lambda_i^{(j+1)} \geq \lambda_i^{(j)} \geq \lambda_{i+1}^{(j+1)}$$

(with $\lambda_i^{(n)} = \lambda_i$). These defining inequalities are organized schematically in the triangular diagram shown in Figure 2.

This polytope comes from the representation theory of the **general linear group** $\mathrm{GL}(n)$. A sequence of integers $\lambda_1 \geq \ldots \geq \lambda_n$ corresponds to an irreducible representation of $\mathrm{GL}(n)$. The lattice points contained in $GZ_n(\lambda_1, \ldots, \lambda_n)$ corresponds to a natural basis of this representation.

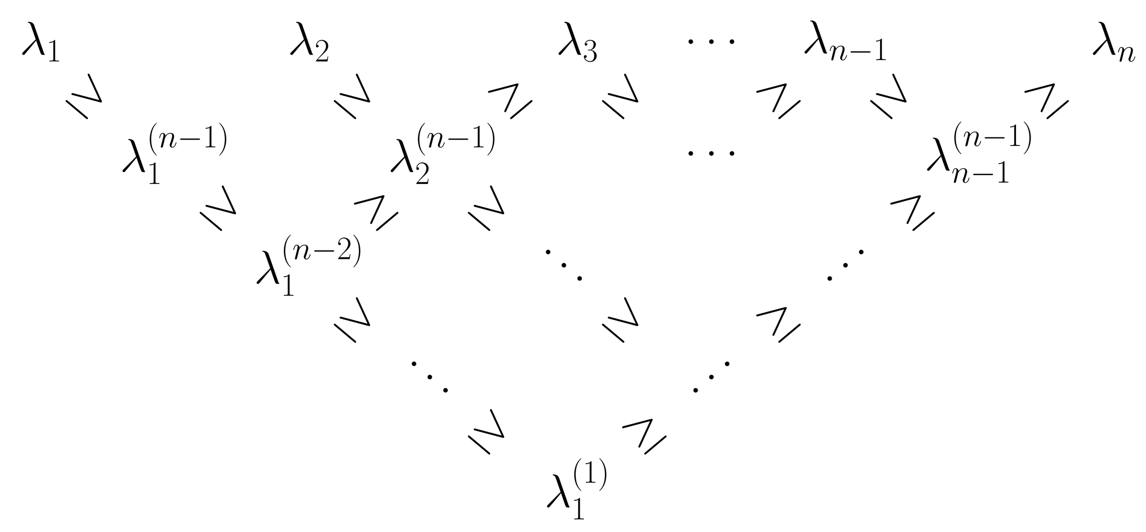

Fig. 2: The defining pattern of the Gelfand-Zetlin polytope.

The Khovanskii-Pukhlikov Theorem

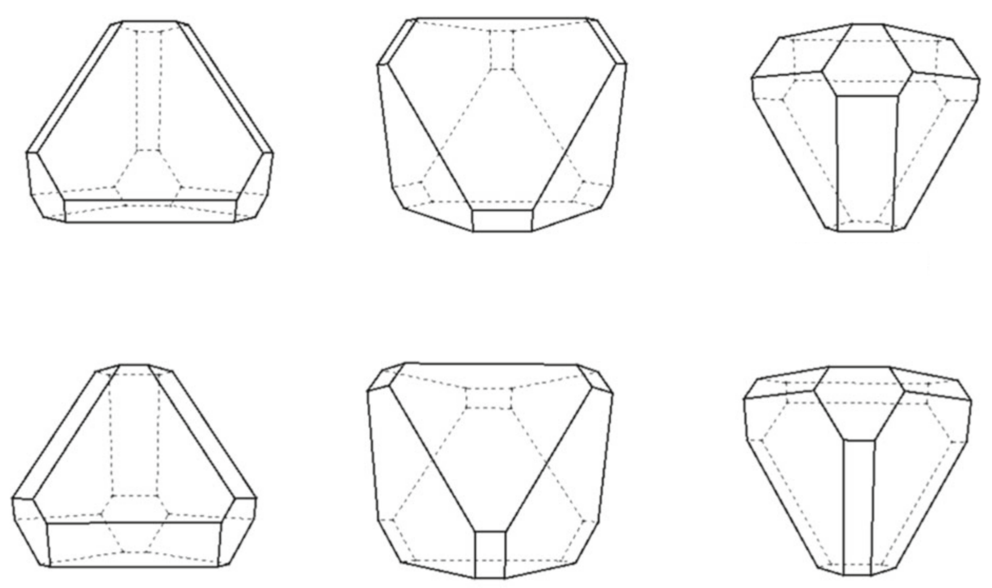

Fig. 3: Analogous polytopes (figure credit to Marzena Szajewska).

There is a theory that corresponds spaces called **toric varieties** with certain "nice" polytopes. In 1992, Khovanskii and Pukhlikov defined the **polytope ring** R_P given by a polytope P and proved that $R_P \cong H^*(X_P)$, where X_P is the toric variety given by P. Later, it was shown that the elements of R_P correspond to faces of P, with multiplication in R_P given by intersections of faces.

The Grassmannians are quotients of a more complicated space, the **complete flag variety** $Fl(n)$, the cohomology of which is also important to Schubert's enumerative geometry. Although $Fl(n)$ is not a toric variety, Khovanskii and Pukhlikov constructed an isomorphism $R_{GZ_n} \cong H^*(Fl(n))$, showing that this polytope ring facilitates computations in Schubert calculus. But on its own, GZ_n is not "nice" enough that R_{GZ_n} may be described by its faces.

Smooth Resolutions

A d-dimensional polytope is said to be **simple** if d edges meet at each vertex. Such polytopes are the best understood: the ring of a simple polytope can be described by the combinatorics of its faces. The 3D GZ-polytope is not simple, as seen in Figure 4 (left). But there is a related simple polytope, called a *smooth resolution*, pictured in Figure 4 (right). The polytope rings of the two are related, so the cohomology ring $H^*(Fl(n)) \cong R_{GZ_n}$ may be studied through the combinatorics of this smooth resolution.

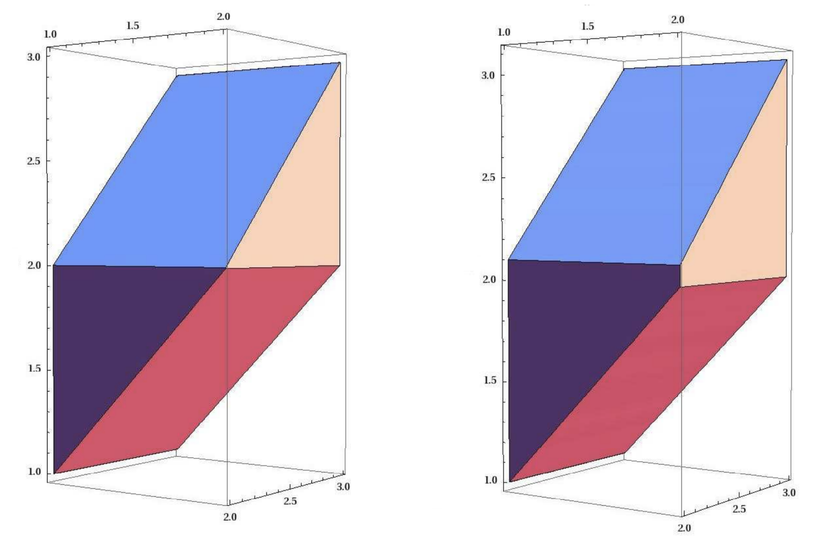

Fig. 4: Resolution of the 3D Gelfand-Zetlin polytope [KST12].

The existence of such a resolution is guaranteed by the theory of toric varieties, but we would like to present a concrete construction, which is useful for computations. Just as with $GZ_n(\lambda)$, the resolution lies in $\mathbb{R}^{\frac{n(n-1)}{2}}$. To define $GZ_n^{\text{res}}(\lambda)$, we replace the inequalities of Figure 2 by

$$\lambda_i^{(j+1)} + \epsilon^{(j)} \geq \lambda_i^{(j)} \geq \lambda_{i+1}^{(j+1)},$$

for fixed $\epsilon^{(1)} > \cdots > \epsilon^{(n-1)}$. These resolutions are combinatorially equivalent to hypercubes, and we conjecture that the corresponding toric variety is comparable to a Hirzebruch surface.

References

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[KST12] V. A. Kiritchenko, E. Yu. Smirnov, and V. A. Timorin. Schubert calculus and Gelfand-Tsetlin polytopes. *Russian Math Surveys*, 67(4(406)):89–128, 2012.

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Further Information

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