Notes on Minimal Surface

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Chapter 1

Preliminary

Absent for the first 20 minutes.
Let $\Sigma \subset (M_n, g)$ be a hypersurface of an n-dimensional Riemannian manifold (M_n, g), where D is the corresponding Riemannian connection. Let $g|_\Sigma$ be the induced metric. Given tangent vectors X, Y of Σ, the second fundamental form II, which is a vector valued symmetric 2-tensor on Σ, is defined as:

$$II(X, Y) = (D_X Y)^\perp.$$

Definition 1.1 (Mean Curvature)**.** *The mean curvature of Σ is defined as the trace of the second fundamental form II, which is*

$$H = \mathrm{tr} II(X, Y) = II(e_i, e_i),$$

where e_i are orthonormal basis of $T\Sigma$.

Proposition 1.2. *Assume M is a n-dimensional manifold, and Σ is a . Let X be the normal on Σ, i.e. $X(p) \perp T_p\Sigma$ for any $p \in \Sigma$, then we have* $\mathrm{div}_\Sigma X = - < X, H >$.

Proof.

$$\begin{aligned} \mathrm{div}_\Sigma X &= < D_{e_i} X, e_i > = - < X, D_{e_i} e_i > \\ &= - < X, (D_{e_i} e_i)^\perp > = - < X, H > \end{aligned}$$

□

Chapter 2

Minimal Graph

Let $\Omega^{n-1} \subset \mathbb{R}^{n-1}$ is a domain, $u : \Omega \mapsto \mathbb{R}^1$ is a smooth function with $|\nabla u| \neq 0$, then, the graph of u, denoted $\Sigma_u = \{(x, u(x)) : x = (x_1, ..., x_{n-1}) \in \Omega\}$. Let $F : \Omega \mapsto \mathbb{R}^n, F(x) = (x, u(x))$ be the C^∞ embedding of $\Sigma_u = F(\Omega)$ to $\mathbb{R}^n$. Then, we are going to compute the volume form of Σ_u with local chart $(F(\Omega), F^{-1})$. Let g be the induced metric of Euclidean Space $\mathbb{R}^n$ on Σ. Hence, we have

$$g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) =< \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} >= \delta_{ij} + \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},$$

i.e.

$$g = I + \nabla u(\nabla u)^T.$$

Moreover, since

$$\begin{aligned} g\nabla u &= (I + \nabla u(\nabla u)^T)\nabla u = \nabla u + \nabla u \nabla u^T \nabla u \\ &= \nabla u + \nabla u(\nabla u^T \nabla u) = \nabla u + |\nabla u|^2 \nabla u \\ &= (1 + |\nabla u|^2)\nabla u \end{aligned} \tag{2.1}$$

we know that $1 + |\nabla u|^2$ is a eigenvalue of g with eigenvector ∇u. Since $\text{Rank}(g - I) = \text{Rank}(\nabla u(\nabla u)^T) = 1$, we know that 1 is also a eigenvalue of g with multiplicity of $n - 2$. Since g is symmetry, we could find $e_2, ..., e_{n-1}$,Hence $\det(g) = 1 + |\nabla u|^2$. Let $dx = dx_1...dx_{n-1}$, we now have volume form $dvol = \sqrt{\det(g)}dx = \sqrt{1 + |\nabla u|^2}dx$.
By now, we can see that

$$\text{Area}(\Sigma_u) = \int_\Omega \sqrt{1 + |\nabla u|^2}dx.$$

Let $\eta \in C_c^\infty(\Omega)$, we then consider Area$(\Sigma_{u+t\eta})$, if the graph Σ_u is minimal, we must have $\frac{d}{dt}\text{Area}(\Sigma_{u+t\eta})|_{t=0} = 0$ for any $\eta \in C_c^\infty(\Omega)$. Hence, we compute

$$\begin{aligned}\frac{d}{dt}\text{Area}(\Sigma_{u+t\eta})|_{t=0} &= \frac{d}{dt}\int_\Omega \sqrt{1+|\nabla(u+t\eta)|^2}dx|_{t=0} = \int_\Omega \frac{d}{dt}\sqrt{1+|\nabla(u+t\eta)|^2}dx|_{t=0} \\ &= \int_\Omega \frac{(\nabla u + t\nabla\eta)\nabla\eta}{\sqrt{1+|\nabla(u+t\eta)|^2}}dx|_{t=0} = \int_\Omega \frac{\nabla u\nabla\eta}{\sqrt{1+|\nabla u|^2}}dx \\ &= -\int_\Omega \eta\text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}),\end{aligned}$$

where the last equality follows from Stoke Formula. Hence, we can see that if Σ_u is minimal graph, we must have $\text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$. We are going to show that $H = \text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})$.

Since

$$\begin{aligned}g(\nabla u(\nabla u)^T) &= (g\nabla u)(\nabla u)^T = ((1+|\nabla u|^2)\nabla u)(\nabla u)^T(\text{By}(2.1)) \\ &= (1+|\nabla u|^2)\nabla u(\nabla u)^T = (1+|\nabla u|^2)(g-I),\end{aligned}$$

we have

$$g((1+|\nabla u|^2)I - \nabla u(\nabla u)^T) = (1+|\nabla u|^2)I.$$

Consequently,

$$(g^{ij}) = g^{-1} = I - \frac{\nabla u(\nabla u)^T}{1+|\nabla u|^2}.$$

It's clear that we can regard Σ_u as lever set given by $h : \mathbb{R}^n \mapsto \mathbb{R}, h(x_1,...,x_n) = x_n - u(x_1,...,x_{n-1})$. Hence $v = \frac{\nabla h}{|\nabla h|} = \frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1)$ is unit normal of Σ_u. So

$$\begin{aligned}H &= g^{ij}II(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = g^{ij}(D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j})^\perp = g^{ij} < D_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}, v > \\ &= \frac{1}{\sqrt{1+|\nabla u|^2}}g^{ij}u_{ij} = \frac{1}{\sqrt{1+|\nabla u|^2}}(\delta^{ij} - \frac{u_iu_j}{1+|\nabla u|^2})u_{ij} \\ &= \frac{u_{ii}}{\sqrt{1+|\nabla u|^2}} - \frac{u_iu_ju_{ij}}{(1+|\nabla u|^2)^{\frac{3}{2}}}\end{aligned}$$

while

$$\text{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = \frac{\partial}{\partial x_i}(\frac{u_i}{\sqrt{1+|\nabla u|^2}}) = \frac{u_{ii}}{\sqrt{1+|\nabla u|^2}} - \frac{u_iu_ju_{ij}}{(1+|\nabla u|^2)^{\frac{3}{2}}}.$$

Hence, we have

Proposition 2.1. $\frac{d}{dt}(\text{Area}\Sigma_{u+t\eta})|_{t=0} = H_{\Sigma_u}$

Chapter 3

The Calibration Properties

3.1 Calibrated property of minimal graphs

We now consider tubular neighbor $O_\Sigma = \cup_{t\in(-\epsilon,\epsilon)}\Sigma_{u+t}$ of Minimal graph Σ_u. We can extend normal field v to $\tilde{v} \in O_\Sigma$, s.t. $\tilde{v}(x_1, ..., x_{n-1}, t) = v(x_1, ..., x_{n-1})$. We claim that:

Proposition 3.1. *Σ_u is minimal graph iff* $\mathrm{div}_{\mathbb{R}^n}\tilde{v} = 0$.

Proof.

$$\begin{aligned}\mathrm{div}_{\mathbb{R}^n}\tilde{v} &= \sum_{i=1}^{n-1}\frac{\partial \tilde{v}^i}{\partial x_i} + \frac{\partial v^n}{\partial t} = \sum_{i=1}^{n-1}\frac{\partial \tilde{v}^i}{\partial x_i} \\ &= \mathrm{div}_{\Sigma_u} v = 0\end{aligned}$$

□

For any $X \in \Gamma(T\mathbb{R}^n)$, we define $i_X : \Omega^* \mapsto \Omega^{*-1}$, by $i_X(\alpha)(*, ..., *) = \alpha(*, ..., *, X)$. Let $\omega = i_{\tilde{v}} dx_1...dx_n$ be the **calibrated form**.

Definition 3.2. *Given any $n-1$ dimensional subspace $V \subset \mathbb{R}^n$, where V is spanned by n-1 orthonormal basis $e_1, ..., e_n$, then define*

$$\omega(V) = \omega(e_1, ..., e_{n-1}).$$

Lemma 3.3. *Assume* $\mathrm{div}_{\mathbb{R}^n}\tilde{v} = 0$*, then we have*
(1)$d\omega = 0$
(2)$|\omega_p(V)| \le 1$ *for any $p \in \Sigma_{u+t}$ and any n-1 dimensional subspace V of $\mathbb{R}^n$, the equality holds when iff $\tilde{v} \perp V$ iff $V \in T_p(\Sigma_{u+t})$.*

Proof. (1)First, we observe that $i_{\tilde{v}}dvol_{\mathbb{R}^n} = \sum_{i=1}^{n-1}(-1)^{i-1}v_i dx_1...\hat{dx_i}...dx_{n-1}$, while $d(i_{\tilde{v}}dvol) = \frac{\partial v}{\partial x_i}dvol_{\mathbb{R}^n} = \mathrm{div}_{\mathbb{R}^n}\tilde{v}dvol_{\mathbb{R}^n} = 0$.
(2)) It's easy to see that

$$dx_1...dx_n(e_1,...,e_{n-1},v) = \det(e_1,...,e_{n-1},\tilde{v})$$

Let $\tilde{v}^{\perp} = P(\tilde{v})$, where $P : \mathbb{R}^n \mapsto V^{\perp}$ is a projection. It's easy to see that $|\tilde{v}^{\perp}| \leq |\tilde{v}|$, and

$$|\det(e_1,e_2,...,e_{n-1},\tilde{v})| = |\det(e_1,e_2,...,e_{n-1},\tilde{v}^{\perp})| \leq 1,$$

while the equality holds iff $|\tilde{v}^{\perp}| = |\tilde{v}|$ iff $\tilde{v} \perp V$. □

Theorem 3.4. *Let Σ_u be a minimal graph defined on $\Omega \in \mathbb{R}^{n-1}$, then*
(1) Σ_u is volume minimizing in $\Omega \times \mathbb{R}$;
(2) If Ω is convex, then Σ_u is volume minimizing in $\mathbb{R}^n$.

Proof. (1) For any $n-1$ dimensional sub manifold $\Sigma_1 \in \Omega \times \mathbb{R}$, with $\partial\Sigma_1 = \partial\Sigma_u$, we can form the n dimensional chain U such that $\Sigma_1 \cap \Sigma_u = \partial U$. Using the Stokes Theorem, we can see

$$\int_{\partial U} w = \int_{\Sigma_1} w - \int_{\Sigma_u} w = \int_U dw = 0,$$

while by Lemma 3.3 (2), we have

$$0 = \int_{\Sigma_1} w - \int_{\Sigma_u} w = \int_{\Sigma_1} w(T_p(\Sigma_1))dvol_{\Sigma_1}(p) - \int_{\Sigma_u} w(T_p(\Sigma_u))dvol_{\Sigma_u}(p)$$
$$\leq \mathrm{Area}(\Sigma_1) - \mathrm{Area}(\Sigma_u)$$

(2) Since Ω is convex, the nearest point projection map $F : \mathbb{R}^n \mapsto \Omega \times \mathbb{R}$ is well defined, and $|\nabla F| \leq 1$ a.e. Hence, for any $\Sigma_1 \in \mathbb{R}^n$, we consider $F(\Sigma_1)$. By (1), we can see that

$$\mathrm{Area}(\Sigma_u) \leq \mathrm{Area}(F(\Sigma_1)) \leq \mathrm{Area}(\Sigma_1),$$

where the last equality follows from $|\nabla F| \leq 1$. □

3.2 A general calibrated argument

Definition 3.5 (Foliation)**.** *We say a domain $\Omega \hookrightarrow (M^n, g)$ in an oriented Riemannian Manifold is foliation (lamination) by hypersurface $\{\Sigma_t, t \in \mathbb{R}\}$ of Ω, if*

(1)Σ_t are disjointed.
(2)$\cup_{t\in\mathbb{R}}\Sigma_t = \Omega$($\cup_{t\in\mathbb{R}}\Sigma_t \subset \Omega$).
(3)$\forall p \in \Omega$, $\exists$ a neighborhood $U_p \in \Omega$, and smooth map $f : U_p \mapsto \mathbb{R}^n$, s.t. $f(\Sigma_t)$ is a set of $\{x_n = t\}$ for all small $t \in \mathbb{R}$.

Theorem 3.6. *Suppose Ω is an open region in an oriented Riemannian manifold M^n, and there exist a foliation of Ω by oriented minimal hyper surfaces, then every leaves of the foliation minimizes volume in Ω.*

Proof. Let $\nu(x)$ be the unit normal vector fields of the foliation, i.e. $\nu \in \Gamma(T\Sigma_t^\perp)$ for some t. Then
Claim: div$\nu = 0$ in Ω if each Σ is minimal.
To prove this, take $\{e_1, \cdots, e_{n-1}\}$ to be tangent orthonormal frames of the foliation, then:

$$\text{div}\nu = \sum_{i=1}^{n-1} < D_{e_i}\nu, e_i > + < D_\nu \nu, \nu >= -H_{\Sigma_t} + 0 = 0.$$

The last two equality follow from $0 = \nu < \nu, \nu >= 2 < D_\nu\nu, \nu >$ and Σ_t is minimal. Define

$$\omega = (-1)^{n-1} i_\nu dvol_M.$$

Using ω as a calibrated form and arguments above, we can show the minimizing property. □

Chapter 4

First Variation

4.1 First Variation Formula

Consider $\Sigma^k \subset M^n$. Let X be a smooth vector field on M with compact support. Let $F_t : M \to M$ be the flow defined by X, i.e.a

$$F_0 = id$$
$$\frac{d}{dt}|_{t=0} F_t(p) = X(p), p \in M.$$

Let $\Sigma_t = F_t(\Sigma)$, the first variation $\delta\Sigma(X) := \frac{d}{dt}\text{Area}(\Sigma_t)|_{t=0}$. Then

Theorem 4.1.

$$\delta\Sigma(X) = \int_\Sigma \text{div}_\Sigma(X) dvol_\Sigma, \tag{4.1}$$

where $\text{div}_\Sigma(X) =< D_{e_i}X, e_i >$, *with* $\{e_1, \cdots, e_k\}$ *an orthonormal basis for* $T\Sigma$.

When Σ is smooth, we can decompose $X = X^T + X^\perp$ to tangent X^T and normal $X^\perp$ parts. So

$$\text{div}_\Sigma(X) = \text{div}_\Sigma(X^T) + \text{div}_\Sigma(X^\perp),$$

where $div_\Sigma(X^\perp) = - < X, H >$. Using the divergence theorem, for general X, we have

$$\delta\Sigma(X) = -\int_\Sigma \langle X, H\rangle d\mu + \int_{\partial\Sigma} \langle X, \eta\rangle d\sigma,$$

where η is the outer normal of $\partial\Sigma$. So we know that

$$\Sigma \text{ is minimal}(\vec{H} = 0) \Longleftrightarrow \delta\Sigma(X) = 0, \ \forall X \text{ of compact support.}$$

Proof. Consider a local parametrization

$$F : \Sigma \times (-\epsilon, \epsilon) \to M,$$

where $F(x,t) = F_t(x)$ with F_t given to be the integration of X above. Let $\{e_1, \cdots, e_k\}$ be an orthnormal basis of $T\Sigma$, $\theta^1, ..., \theta^k$ be the dual basis, $(g_t)_{ij} = (F_t)^* g(e_i, e_j)$ then

$$\begin{aligned} \frac{d}{dt}(g_t)_{ij}|_{t=0} &= (L_X g)(e_i, e_j) = Xg(e_i, e_j) - g([X, e_i], e_j) - g(e_i, [X, e_j]) \\ &= g(D_{e_i}X, e_j) + g(D_{e_i}X, e_j) - g(D_X e_i, e_j) - g(e_j, D_X e_i) \\ &= g(D_{e_i}X, e_j) + g(D_{e_i}X, e_j) \end{aligned}$$

Hence $\mathrm{tr}(\frac{d}{dt}(g_t)_{ij}|_{t=0}) = 2g(D_{e_i}X, e_i) = 2\mathrm{div}_\Sigma X$.
Moreover, we have

$$\frac{d}{dt}\det(g_t)|_{t=0} = \mathrm{tr}(\frac{d}{dt}((g_t)_{ij})|_{t=0}) = 2\mathrm{div}_\Sigma(X) \tag{4.2}$$

$$\frac{d}{dt}\mathrm{Area}(\Sigma_t)|_{t=0} = \int_\Sigma \frac{d}{dt}\sqrt{\det g(t)}|_{t=0}\theta^1 ... \theta^k$$

$$= \int_\Sigma \frac{\frac{d}{dt}\det(g_t)|_{t=0}}{2}\theta^1 ... \theta^k = \int_\Sigma \mathrm{div}_\Sigma X \theta^1 ... \theta^k$$

□

4.2 Examples

Definition 4.2. *Let $\Sigma^k \subset (M^n, g)$, then Σ is called minimal iff $H = 0$.*

Example 4.3. *(1)$\mathbb{R}^2 \subset \mathbb{R}^3$ is minimal.*
(2) The Helicoid $H \subset \mathbb{R}^3$ is minimal, where $H = \{(t\cos s, t\sin s, s)\}$, i.e., H is minimal graph defined by $u(x_1, x_2) = \arctan(\frac{x_2}{x_1})$. It's straightforward to check that

$$\mathrm{div}(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0.$$

(3) The Catenoid C, which is $C = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = (\cosh x_3)^2\}$ is minimal. In addition, we could see that the surface is scale invariant, then let $\lambda C := \{(x_1, x_2, x_3) :\}$

4.3 Convex hull property

Consider $i : \Sigma^k \hookrightarrow \mathbb{R}^n$ be the C^∞ embedding. Let $x = (x_1, \cdots, x_n)$ be coordinates on $\mathbb{R}^n$, then we have:

Proposition 4.4.

$$\Delta_\Sigma x|_\Sigma = H$$

Proof. Let $\{e_1, \cdots, e_k\}$ be a local orthonormal basis for $T\Sigma$, then

$$\Delta_\Sigma x^i = \sum_{j=1}^{k}(e_j e_j x^i - (\nabla^\Sigma_{e_j} e_j)x^i) = \sum_{j=1}^{k}(\nabla^\perp_{e_j} e_j)x^i.$$

As $e_j x = e_j$ and $\sum \nabla^\perp_{e_j} e_j = H$, $\Delta_\Sigma x = H$. □

Corollary 4.5. *if Σ is minimal, $\Delta_\Sigma x = 0$.*

Definition 4.6 (Convex Hall)**.** *For any subset $X \in \mathbb{R}^n$, we defined its convex hall $\mathcal{C}(X)$ by*

$$\mathcal{C}(X) = \cap\{H : H \textit{ is the half space of } \mathbb{R}^n \textit{ contains } X\}$$

Corollary 4.7. *If $(\Sigma^k, \partial\Sigma)$ is compact minimal in $\mathbb{R}^n$, then $\Sigma \subset \mathcal{C}(\partial\Sigma)$.*

Proof. We just need to prove for any Half space H also contains $\partial\Sigma$, H contains Σ. Assume $H = \{x \in \mathbb{R}^n : l(x) \le a, v \in \mathbb{R}^n, a \in \mathbb{R}\}$ for some linear function l. Now restrict l on Σ. Since l is linear, $\Delta_\Sigma l(x) = l(\Delta_\Sigma x) = 0$, which mean l is harmonic on Σ. Moreover, since $l|_{\partial\Sigma} \le a$, by weak maximal principle(Lemma 4.8), we have $l_\Sigma \le a$, which mean $\Sigma \subset H$. □

The following lemma is a standard result in EPDE:

Lemma 4.8. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain. For $f \in C^2(\Omega) \cap C^0(\Omega)$ satisfying $\Delta f \le 0$, we have $\sup_\Omega f = \sup_{\partial\Omega} f$.*

4.4 Fluxes

In the case of a minimal sub manifold $\Sigma^k \subset \mathbb{R}^n$,

$$\Delta_\Sigma x_i = 0 \Leftrightarrow (*d*d + d*d*)x_i = 0 \Leftrightarrow d*dx = 0 \Leftrightarrow *dx_i \text{ is closed.}$$

Hence $*dx_i$ defines a $(k-1)$ dimensional deRham cohomology class. So for any $k-1$ cycle $\Gamma^{k-1} \subset \Sigma^k$, we can define

$$F([\Gamma]) = \int_\Gamma *dx_i = \int_\Gamma \langle \frac{\partial}{\partial x_i}, \eta \rangle d\sigma,$$

where η is the unit outer normal of Γ. The second equality follows from the fact that $*dx_i = i_{\frac{\partial}{\partial x_i}} dv_\Sigma = \langle \frac{\partial}{\partial x_i}, \eta \rangle d\sigma$, with $i_* dv$ the inner multiplication.

Proposition 4.9. *F depends on the homotopy class of Γ.*

Proof. Assume Γ_1 and G_2 are homotopic to each other, then there exist k-cycle T, s.t. $\partial T = \Gamma_1 - \Gamma - 2$. Then we have

$$\int_{\Gamma_1} *dx_i - \int_{\Gamma_2} *dx_i = \int_{\partial T} *dx_i = \int_T d * dx_i = 0.$$

□

Hence we have a group homeomorphism:

$$F : H_{k-1}(\Sigma, \mathbb{Z}) \to \mathbb{R}.$$

For the general cases, if $\Sigma^k \subset (M^n, g)$ is a minimal sub manifold, i.e. $H = 0$, and X a Killing vector field on M, i.e. $L_X g = 0$, then there exists a homeomorphism:

$$F_X : H_{k-1}(\Sigma, \mathbb{Z}) \to \mathbb{R}.$$

Proof. Let $V = X^T$ the tangential part of X on Σ, then

$$div_\Sigma(V) = \sum_{i=1}^k \langle D_{e_i}(X - X^\perp), e_i \rangle = \sum_{i=1}^k \langle D_{e_i} X, e_i \rangle - \sum_{i=1}^k \langle D_{e_i} X^\perp, e_i \rangle = 0.$$

For the last equality, we have

$$\langle D_{e_i} X^\perp, e_i \rangle = -\langle X^\perp, D_{e_i} e_i \rangle = H(X) = 0,$$

and since

$$0 = L_X g(e_i, e_i) = 2\langle [X, e_i], e_i \rangle = \langle D_X e_i, e_i \rangle - \langle D_{e_i} X, e_i, \rangle$$

we know that

$$\langle D_{e_i} X, e_i \rangle = \langle D_X e_i, e_i \rangle = 1/2 X \langle e_i, e_i \rangle = 0.$$

Let $\omega = i_V dvol_\Sigma$, then ω is a closed $k-1$ form on Σ, since $d(i_V dvol) = \mathrm{div} V dvol$. Hence we can define the flux as above. □

Chapter 5

Monotonicity formula and Bernstein Theorem

5.1 Monotonicity formula

Fix $x_0 \in \Sigma$, let $s, t > 0$ small enough s.t. $B_s(x_0)\partial\Sigma = \emptyset, B_r(x_0)\partial\Sigma = \emptyset$, where $B_t(x_0)$ is a ball of radius t with center x_0 in $\mathbb{R}^n$.

Theorem 5.1. *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal surface, then*

$$\frac{\text{Area}(B_t(x_0) \cap \Sigma)}{t^k} - \frac{\text{Area}(B_s(x_0) \cap \Sigma)}{s^k} = \int_{\Sigma \cap (B_t(x_0) \setminus B_s(x_0))} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}} dvol_\Sigma,$$

where $(x - x_0)^\perp$ is the projection to the normal part of Σ of $(x - x_0)^\perp$.

We need the co-area formula before the proof.

Lemma 5.2. *(Co-area Formula) Let $h : \Sigma \to \mathbb{R}_+$ be a nonnegative Lipschitz function on a Riemannian manifold Σ, and proper i.e. $\{x \in \Sigma : \ h(x) \leq a\}$ is compact for all a. Given f integrable on Σ, then*

$$\int_{h \leq t} f|\nabla_\Sigma h| = \int_{-\infty}^{t} (\int_{h=t} f) d\tau.$$

Remark 5.3. *This follows heuristically from the ideas that $dv_\Sigma = \frac{dt \wedge dv_{\{h=t\}}}{|\nabla_\Sigma h|}$ when t is a regular value and Fubini Theorem.*

Proof. (Monotonicity formula)Take $h(x) = |x - x_0|$, then $\{h \leq t\} = B_t(x_0)$. Let $X = x - x_0$, then $div_\Sigma(X) = \sum_{i=1}^{k} \nabla_{e_i} X \cdot e_i = \sum_{i=1}^{k} e_i \cdot e_i = k$, where

$\{e_1, \cdots, e_k\}$ is an orthornormal basis on Σ. Then by the first variation formula 4.1,

$$\delta\Sigma_{B_r(x_0)}(X) = \int_{\Sigma\cap B_r(x_0)} div_{\Sigma_0}(X) = \int_{\Sigma\cap\partial B_r(x_0)} X\cdot\eta,$$

where η is the co-normal vector of $\Sigma\cap\partial B_r(x_0)$, and $\eta = \frac{\nabla^\Sigma|x-x_0|}{|\nabla^\Sigma|x-x_0||} = \frac{(x-x_0)^T}{|(x-x_0)^T|}$. Using the Co-area formular,

$$\begin{aligned} k|\Sigma\cap B_r(x_0)| &= \int_{\Sigma\cap\{|x-x_0|=r\}} |(x-x_0)^T| = r\int_{\Sigma\cap\{|x-x_0|=r\}} \frac{|(x-x_0)^T|}{|x-x_0|} \\ &= r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^T|}{|x-x_0|}|\nabla^\Sigma|x-x_0|| = r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^T|^2}{|x-x_0|^2} \\ &= r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} (1-\frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}) \\ &= r\frac{d}{dr}|\Sigma\cap B_r(x_0)| - r\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}. \end{aligned}$$

Multiplying the above by r^{-k-1}, we can re-write it as,

$$\begin{aligned} \frac{d}{dr}(r^{-k}|\Sigma\cap B_r(x_0)|) &= r^{-k}\frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}, \\ &= \frac{d}{dr}\int_{\Sigma\cap B_r(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}}. \end{aligned}$$

In the last step, we can use the co-area formula again to absorb the factor r^{-k} into the integration. So we can get the monotonicity formula by integrating the above equation. □

Corollary 5.4. *Let Σ^k be a smooth minimal surface in $\mathbb{R}^n$, with boundary $\Sigma\cap\partial B_R(0)$ inside the ball $B_R(0)$. If $x_0\in\Sigma\cap B_R(0)$, and $\sigma < R-|x_0|$, then*

$$\omega_k\sigma^k \le \text{Area}(\Sigma\cap B_\sigma(x_0)) \le \frac{\sigma^k}{(R-|x_0|^k)}\text{Area}(\Sigma\cap B_R(0)),$$

where ω_k is the volume of unit ball $B_1^k(0)$ in $\mathbb{R}^k$.

Proof. The first inequality comes from the Monotonicity formula while comparing $B_\sigma(x_0)$ with an arbitrary small ball $B_r(x_0)$, with $\lim_{r\to 0} r^{-k}|\Sigma\cap B_r(x_0)| = \omega_k$, when $x_0\in\Sigma$ and Σ smooth. The second inequality is a direct consequence of the Monotonicity formula while comparing $B_\sigma(x_0)$ with a large ball $B_r(x_0)$ exhausting the whole $B_R(0)$. □

Definition 5.5. *The **density** of Σ at x_0 is defined as:*

$$\Theta_{x_0} = \lim_{r\to 0}(\omega_k r^k)^{-1}|\Sigma \cap B_r(x_0)|.$$

Example 5.6.

5.2 Bernstein's theorem (n=2)

Theorem 5.7 (S. Bernstein (1912))**.** *Given a minimal graph $\Sigma^2 \subset \mathbb{R}^3$, $\Sigma = \{(x, u(x)) : x \in \mathbb{R}^2\}$. If u is defined on all of $\mathbb{R}^2$, then u is a linear function, and Σ is a plane.*

Theorem 5.8 (Bernstein's Big Theorem:)**.** 1° *PDE version: let $u \in C^2(\mathbb{R}^2)$ and $\sum_{i,j=1}^2 a_{ij}u_iu_j = 0$, with $(a_{ij}) > 0$. If u is bounded, then $u \equiv const$;*
2° : *$\Sigma^2 - Graph_u$, where u is defined on R^2 and bounded, if the Gaussian curvature $K_\Sigma \leq 0$, then Σ is a plane.*

Consider the Gauss Maps:

$$N : \Sigma^2 \to S^2,$$

where N maps a point to the unit normal vector at that point.

Lemma 5.9. *If $\vec{H} = 0$, then N is a conformal and orientation reversing map, i.e. $\forall v, w \in T_x\Sigma$, if $v \cdot w = 0$ and $|v| = |w|$, then $\nabla_v N \cdot \nabla_w N = 0$, and $|\nabla_v N| = |\nabla_w N|$. Furthermore $|\nabla_v N| \leq \frac{1}{\sqrt{2}}|A||v|$, and $N^*(\omega_{S^2}) = K_\Sigma \omega_\Sigma = -\frac{1}{2}|A|^2\omega_\Sigma$.*

Proof. We only need to check that under principle vector fields. Take principle vector fields $\{e_1, e_2\}$ for Σ, i.e. $\nabla_{e_1} N = -K_1 e_1$, $\nabla_{e_2} N = -K_2 e_2$. So $|\nabla_{e_1} N| = |K_1| = |K_2| = |\nabla_{e_2} N|$, by the minimality $H = K_1 + K_2 = 0$. Hence $|\nabla_v N| \leq |K||v| = \frac{1}{\sqrt{2}}|A|$. Furthermore, the Jacobian of N is $Jac(N) = K_1K_2 = -\frac{1}{2}|A|^2$. □

Remark 5.10. *We are going to prove Bernstein's theorem based on the following two facts:*
(1) We could choose an coordinate system and orientation of Σ, s.t. the image of Gauss map lie in S^2_+.
(2) By Theorem 3.4(2), Σ_u is area minimizing.

To apply fact (1), we have:

Proposition 5.11. *Given a minimal $\Sigma^2 \subset \mathbb{R}^3$, with the image of the Gauss Maps lying in the upper hemisphere $N(\Sigma) \subset S^2_+$, if φ has compact support on Σ, then there exists a constant $C > 0$, such that*

$$\int_\Sigma |A|^2\varphi^2 \leq C \int_\Sigma |\nabla\varphi|^2.$$

Proof. Since S^2_+ is simply connected, the closed form $\omega_{S^2} = d\alpha$ is also exact. Hence

$$-\frac{|A|^2}{2}\omega_\Sigma = N^*\omega_{S^2} = d(N^*\alpha).$$

So

$$\int_\Sigma |A|^2\varphi^2\omega_\Sigma = -2\int_\Sigma \varphi^2 d(N^*\alpha) = 4\int_\Sigma \varphi d\varphi \wedge N^*\alpha$$
$$\leq 4\int_\Sigma |\varphi||\nabla\varphi||N^*\alpha|\omega_\Sigma.$$

Since $|N^*\alpha| \leq |A||\alpha| \leq C|A|$,

$$\int_\Sigma |A|^2\varphi^2 \leq C\int_\Sigma (|\varphi||A|)(|\nabla\varphi|) \leq \frac{C\epsilon}{2}\int |A|^2\varphi^2 + \frac{C}{2\epsilon}\int |\nabla\varphi|^2.$$

Choose $\epsilon > 0$ so that $C\epsilon = 1/2$, we get the inequality. □

To apply fact (2), we have

Proposition 5.12. *Let Σ be an entire minimal graph, then $|\Sigma \cap B_R(0)| \leq 4\pi R^2$, $\forall R > 0$.*

Proof. This comes from the area-minimizing property of minimal graphs. We can compare $|\Sigma \cap B_R(0)|$ with the large area of the truncated surfaces of $B_R(0)$ by Σ. □

From Proposition 5.11, if we could find a suitable cut-off function φ, s.t. $\varphi = 1$ on B_R and vanish outside B_{2R}. Moreover, it makes right hand side of equality in Proposition 5.11 goes to 0 as R goes to ∞. Then, we must have $A = 0$.

Lemma 5.13. *When $\Sigma = Graph_u$ and u is an entire function on $\mathbb{R}^2$, then we can choose a Lipschetz $\varphi = \varphi_R$, such that $\int_\Sigma |\nabla\varphi|^2 \to 0$ as $R \to \infty$.*

Proof. Now choose

$$\varphi_R(r) = \begin{cases} 1 & \text{if } r \leq R \\ 1 - \log(r/R)/\log R & \text{if } R < r < R^2 \\ 0 & \text{if } r \geq R^2 \end{cases}$$

where r is the distance function of $\mathbb{R}^3$. By discretize $B_{R^2} \backslash B_R = \cup_{k=1}^{\log R}(B_{e^k R} \backslash B_{e^{k-1}R})$, we have

$$\int_\Sigma |\nabla\varphi|^2 \omega_\Sigma = \frac{1}{(\log R)^2} \int_{\Sigma \cap (B_{R^2} \backslash B_R)} \frac{1}{r^2} \omega_\Sigma = \frac{1}{(\log R)^2} \sum_{k=1}^{\log R} \int_{\Sigma \cap (B_{e^k R} \backslash B_{e^{k-1}R})} \frac{1}{r^2} \omega_\Sigma$$

$$\leq \frac{1}{(\log R)^2} \sum_{k=1}^{\log R} \frac{1}{(e^{k-1}R)^2} |\Sigma \cap (B_{e^k R} \backslash B_{e^{k-1}R})| \leq \frac{1}{(\log R)^2} \sum_{k=1}^{\log R} \frac{1}{(e^{k-1}R)^2} C(e^k R)^2$$

$$= \frac{C}{(\log R)^2} \sum_{k=1}^{\log R} \frac{1}{e^2} = \frac{C}{\log R} \to 0,$$

where in the second " $\leq$ ", we used the quadratic area bound Lemma above. □

Bernstein's Theorem. When $\Sigma = Graph_u$ is an entire graph, the image of the Gauss Maps $N(\Sigma)$ lies in an hemisphere, so we get $\int_\Sigma |A|^2 \varphi^2 \leq C \int_\Sigma |\nabla\varphi|^2$. Then if we take the φ_R in the above Lemma, and let $R \to \infty$, we see that $\int_\Sigma |A|^2 \to 0$. So $A = 0$, and Σ is a plane. □

Chapter 6

Second variation and Stability

6.1 Second Variation of Volume

Consider a minimal $\Sigma^k \subset M^n$, i.e. $\vec{H} = 0$. Given a vector field X on Σ, let F_t be the geometric flow related to X, i.e.

$$F_0 = id$$
$$\frac{d}{dt} F_t(p)|_{t=0} = X(p), p \in M$$

Theorem 6.1 (Second Variation Formula)**.**

$$\delta^2\Sigma(X,X) := \frac{d^2}{dt^2}|_{t=0}\text{Area}(\Sigma_t) = \int_\Sigma [|D^\perp X|^2 - |\langle \vec{A}, X\rangle|^2 - \sum_{i=1}^{k} R^M(e_i, X, e_i, X)], \tag{6.1}$$

where $\{e_1, \cdots, e_k\}$ is an orthonormal basis tangent to Σ, and X is compact supported and normal on Si.

Proof.
- $F : \Sigma \times (-\epsilon, \epsilon) \to M$, with $\{x^1, \cdots, x^k\}$ local coordinates on Σ, s.t. $F(p,t) = F_t(p), p \in \Sigma$. Then $dvol_t = \sqrt{detg(t)}dx$, where $g_{ij}(t) = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle$.
- $$\frac{d}{dt}\sqrt{detg(t)} = \frac{1}{2} g^{ij} \dot{g}_{ij} \sqrt{detg(t)}$$

- $$\frac{d^2}{dt^2}\sqrt{detg(t)} = \frac{1}{4}(g^{ij}\dot{g}_{ij})^2\sqrt{detg} + \frac{1}{2}g^{ij}\ddot{g}_{ij}\sqrt{detg} + \frac{1}{2}(\dot{g}^{ij}\dot{g}_{ij})\sqrt{detg},$$

  where $\dot{g}^{ij} = -g^{ik}g^{jl}\dot{g}_{kl}$.

- $$\begin{aligned}\dot{g}_{ij} &= \langle D_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\rangle + \langle \frac{\partial F}{\partial x^j}, D_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^i}\rangle \\ &= \langle D_{\frac{\partial F}{\partial x_i}}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}\rangle + \langle \frac{\partial F}{\partial x^j}, D_{\frac{\partial F}{\partial x_i}}\frac{\partial F}{\partial t}\rangle \\ &= -\langle \frac{\partial F}{\partial t}, D_{\frac{\partial F}{\partial x_i}}\frac{\partial F}{\partial x^j}\rangle - \langle D_{\frac{\partial F}{\partial x_i}}\frac{\partial F}{\partial x^j}, \frac{\partial F}{\partial t}\rangle \\ &= -\langle \vec{A}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), X\rangle . (\text{When } t = 0.)\end{aligned}$$

  So
  $$(g^{ij}\dot{g}_{ij})|_{t=0} = -2\langle \vec{H}, X\rangle = 0,$$
  and
  $$(\dot{g}^{ij}\dot{g}_{ij})|_{t=0} = -4|\langle \vec{A}, X\rangle|^2.$$

- $$\begin{aligned}\ddot{g}_{ij} &= \langle D_{\frac{\partial F}{\partial t}}D_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}\rangle + \langle D_{\frac{\partial F}{\partial x^j}}\frac{\partial F}{\partial t}, D_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^i}\rangle \\ &+ \langle D_{\frac{\partial F}{\partial t}}D_{\frac{\partial F}{\partial x^j}}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^i}\rangle + \langle D_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial t}, D_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^j}\rangle \\ &= \langle R^M(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^i})\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}\rangle + \langle D_{\frac{\partial F}{\partial x^i}}\ddot{F}, \frac{\partial F}{\partial x^j}\rangle + \langle D_{\frac{\partial F}{\partial x^i}}X, D_{\frac{\partial F}{\partial x^j}}X\rangle \\ &+ \langle R^M(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j})\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^i}\rangle + \langle D_{\frac{\partial F}{\partial x^j}}\ddot{F}, \frac{\partial F}{\partial x^i}\rangle + \langle D_{\frac{\partial F}{\partial x^j}}X, D_{\frac{\partial F}{\partial x^i}}X\rangle \\ &= -R^M(X, \frac{\partial}{\partial x^i}, X, \frac{\partial}{\partial x^j}) + \langle D_{\frac{\partial}{\partial x^i}}\ddot{F}, \frac{\partial}{\partial x^j}\rangle + \langle D_{\frac{\partial}{\partial x^i}}X, D_{\frac{\partial}{\partial x^j}}X\rangle \\ &- R^M(X, \frac{\partial}{\partial x^j}, X, \frac{\partial}{\partial x^i}) + \langle D_{\frac{\partial}{\partial x^j}}\ddot{F}, \frac{\partial}{\partial x^i}\rangle + \langle D_{\frac{\partial}{\partial x^j}}X, D_{\frac{\partial}{\partial x^i}}X\rangle\end{aligned}$$

  So
  $$(g^{ij}\ddot{g}_{ij})|_{t=0} = -2g^{ij}R^M(X, \frac{\partial}{\partial x^i}, X, \frac{\partial}{\partial x^j}) + 2div_\Sigma \ddot{F} + 2|D^\perp X|^2 + \underbrace{2g^{ij}\langle D^T_{\frac{\partial}{\partial x^i}}X, D^T_{\frac{\partial}{\partial x^j}}X\rangle}_{=2|\langle \vec{A}, X\rangle|^2}$$

- Combining all the above,

$$\frac{d^2}{dt^2}|_{t=0}\sqrt{detg(t)} = div_\Sigma \ddot{F} + |D^\perp X|^2 - |\langle \vec{A}, X\rangle|^2 - g^{ij}R^M(X, \partial x^i, X, \partial x^j).$$

An integration on Σ finishes the proof. □

Theorem 6.2. *In the case $\Sigma^{n-1} \subset M^n$ is a hyper surface and 2-sided(there exist global unit normal ν), $X = \varphi\nu$, with φ a function with compact support, then*

$$\delta^2\Sigma(\varphi,\varphi) = \int_\Sigma [|\nabla\varphi|^2 - (|A|^2 + Ric^M(\nu,\nu))\varphi^2] \tag{6.2}$$

Proof. We have

$$|D^\perp X| = \langle D^\perp X, \nu\rangle = \langle \nabla\phi\nu, \nu\rangle + \langle \phi D^\perp v, v\rangle = |\nabla\phi|,$$

$$|\langle A, X\rangle|^2 = |\phi|^2|\langle A, v\rangle|^2 = \phi^2|A|^2,$$

$$\sum_{i=1}^{n-1} R(X, e_i, X, e_i) = \sum_{i=1}^{n-1}\phi^2 R(\nu, e_i, \nu, e_i) = \phi^2 Ric(\nu,\nu).$$

□

6.2 Jacobi operator and Stability

Definition 6.3 (Stability)**.** *Σ is stable if $I(X,X) := \delta\delta\Sigma(X,X) \geq 0$, $\forall X$ normal with compact support.*

Corollary 6.4. *If $Ric \geq 0$, then there is no closed 2-sided stable minimal hypersurface.*

Proof. Take $\phi = 1$ in Theorem 6.2. □

Corollary 6.5. *If $\Sigma^{n-1} \subset M^n$ is two-sided and stable minimal surface, then $Ric(\nu,\nu) = 0$ on Σ, where ν is normal on Σ. Moreover, Σ is total geodesic.*

Proof. Take $\phi = 1$. □

Remark 6.6. *The 2-side condition is essential. Since $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$ is stable, but $Ric \neq 0$.*

Proof. □

Corollary 6.7. *Assume $\Sigma \hookrightarrow M^3$ is a closed, two-sided stable minimal hypersurface with scale curvature $R \geq 0$. Then the genus g of Σ must be 0 or 1.*

Example 6.8. *(1)For $g = 0$, we have $S^2 \hookrightarrow S^2 \times S^1$ with $R = 1$.*
(2)For $g = 1$, we have $\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$, with $R = 0$.

In order to prove Corollary 6.7, we need

Lemma 6.9. *Let $\Sigma^{n-1} \hookrightarrow M^n$, ν is normal on Σ locally, then we have*

$$|A|^2 + Ric^M(\nu) = \frac{1}{2}(R^M - R^\Sigma + |A|^2).$$

Proof of Corollary 6.7. First, we have

$$\begin{aligned}\int |\nabla\phi|^2 &= \int (|A|^2 + Ric(\nu,\nu))\phi^2 \\ &= \frac{1}{2}\int (R^M - R^\Sigma + |A|^2)\phi^2 \\ &\geq \frac{1}{2}\int -R^\Sigma\phi^2 = \frac{1}{2}\int -K\phi^2.\end{aligned}$$

Take $\phi = 1$, by Gauss-Bonnet, we have $0 \leq 4\pi(2 - 2g)$. Hence, we know that $g = 0, 1$. □

Proof of Lemma 6.9. Locally, let $e_1, ...e_{n-1}$ be a orthonormal basis of Σ. Let $R^\Sigma_{ijij} = R^\Sigma(e_i, e_j, e_i, e_j), R^M_{ijij} = R^M(e_i, e_j, e_i, e_j), h_{ij} = \langle D_{e_i}e_j, \nu\rangle$. Then we have

$$R^\Sigma_{ijij} + h^2_{ij} - h_{ii}h_{jj} = R^M_{ijij}.$$

Sum over i, j, we have $R^\Sigma + |A|^2 - H^2 = R^M - 2Ric(v, v)$ □

We now consider hypersurface case, i.e. $k = n - 1$, $X = \varphi\nu$,

$$\begin{aligned}I(\varphi,\varphi) \equiv \delta^2\Sigma(\varphi,\varphi) &= \int_\Sigma [|\nabla\varphi|^2 - (|A|^2 + Ric^M(\nu,\nu))\varphi^2] \\ &= \int_\Sigma -\Delta\varphi\phi - (|A|^2 + Ric^M(\nu,\nu))\varphi^2] \\ &= -\int_\Sigma \varphi L\varphi,\end{aligned}$$

where the Jacobi operator L is

$$L\varphi = \Delta\varphi + (\underbrace{|A|^2 + Ric^M(\nu,\nu)}_{Q})\varphi. \tag{6.3}$$

When boundary exists $(\Sigma, \partial\Sigma)$, L has discrete eigenvalues λ_j and eigenfunctions u_j with Dirichlet condition, i.e.

$$\begin{aligned} Lu_j + \lambda_j u_j &= 0, \text{in } \Sigma \\ u_j &= 0, \text{on } \partial\Sigma \end{aligned}$$

and

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

with $\lambda_n \to +\infty$.

Definition 6.10. *Morse Index of Σ is defined as the number of negative eigenvalues counted with multiplicity.*

Proposition 6.11. *(1)λ_1 has multiplicity 1;*
(2)If u_1 is an eigenfunction of λ_1, u_1 does not change sign.

Proof. We just prove (2) Here. By the variational characterization, u_1 minimizes $I(\varphi, \varphi)$ among all φ with $\varphi \equiv 0$ on $\partial\Sigma$ and $\int_\Sigma \varphi^2 = 1$. Since $|(|\varphi|, |\varphi|) = I(\varphi, \varphi)$, if u_1 is the first eigenfunction, so is $|u_1|$. So $u_1 = |u_1|$, or there is a contradiction to $u_1 \in C^\infty$. The fact that u_1 does not change sign shows that the dimension of the eigen space of λ_1 is 1, or we can always form some eigenfunction changing sign. □

Definition 6.12. *When Σ is non-compact, then*

$$Ind(\Sigma) = \lim_{i \to \infty} Ind(\Omega_i),$$

where $\{\Omega_i\}_{i=1}^\infty$ is an open exhaustion of Σ, i.e. $\Sigma = \cup_{i=1}^\infty \Omega_i$, $\overline{\Omega_i} \subset \Omega_{i+1}$, with $\partial\Omega_i$ smooth and compact.

Remark 6.13. *In fact, the definition is independent of the exhaustion, say $\{\Omega_i\}$ and $\{\tilde{\Omega}_i\}$.*

Proof. First, we have $\lambda_k = inf\{\lambda(V) : V \subset C_c^\infty(\Omega), dim(V) = k\}$, where $\lambda(V) = sup\{-\int Lff : f \in V, \int |f|^2 = 1\}$(cf. [1] section 4.5). By this we can see $Ind(\Omega)$ is non decreasing when Ω is expanding (since $C_c^\infty(\Omega) \subset C_c^\infty(\Omega')$ when $\Omega \subset \Omega'$), so we can always embed $\Omega_i \subset \tilde{\Omega}_{i'}$ for $i' \gg i$, so $Ind(\Omega_i) \leq Ind(\tilde{\Omega}_{i'})$, and $\lim_{i \to \infty} Ind(\Omega_i) \leq \lim_{i' \to \infty} Ind(\tilde{\Omega}_{i'})$, and vise versa. □

When Σ is open:

$$\lambda_1(\Sigma) = \lim_{i \to \infty} \lambda_1(\Omega_i) \in [-\infty, \lambda_1(\Omega_1)),$$

Σ is stable if $\lambda_1(\Sigma) \geq 0$, or equivalently $\lambda_1(\Omega) \geq 0$ for all $\overline{\Omega} \subset \Sigma$.

Remark 6.14. *By the variational characterization, $\lambda_1(\Omega)$ is strictly decreasing as Ω is expanding, so we can argue as above to show the well-definedness of $\lambda_1(\Sigma)$.*

Chapter 7

Additional Topics in Stability I

- Let $\Sigma^2 \subset M^3$ be a closed, two-sided stable minimal hypersurface with positive scale curvature R. In the proof Corollary 6.7, we know that if the scale curvature $R > 0$, we must have $\Sigma = (S^2, g)$. Hence, we have the following positive mass theorem

  Theorem 7.1. *There is no positive scale metric on* $\mathbb{T}^3$.

  Proof. By an Theorem 4.1 in [6] given by Schoen-Yau, there exists an area-minimizing immersion $u : \mathbb{T}^2 \hookrightarrow \mathbb{T}^3$. If there exists a positive scale metric on $\mathbb{T}^3$, then we know from the proof of Corollary 6.7 that the genus of $u(\mathbb{T})$ must be 0, which is a contradiction. □

- We now focus on the 3-manifold M, which admits a scale curvature $R \geq 6$ in the following. Marques-Neves(2013) and Song(2015) (??? Not sure) show that if there exists a least area minimal surface of $\Sigma \subset M$, then Area$(\Sigma) \leq 4\pi$. The equality holds iff M is the standard 3-sphere S^3.
  (1)We would like to study the stable 1-sided minimal surfaces first. An example is that $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$.
  Let $\Sigma^2 \subset M^3$ be a stable 1-side minimal surface, X be a normal field on Σ with compact support. Moreover, let $\pi : \tilde{\Sigma} \mapsto \Sigma$ be a 2-cover of Σ, $\tilde{\nu}$ be a unit normal vector field of $\tilde{\Sigma} \hookrightarrow M^3$, $i : \tilde{\Sigma} \mapsto \tilde{\Sigma}$ be the deck transformation. Then we have
  (a)$\tilde{\nu} \circ i = -\tilde{\nu}$.

(b)Let $\tilde{X} = X \circ \pi$, then $\tilde{X} \circ i = X$.
Hence, if we assume $\tilde{X} = \phi\tilde{\nu}$, then we have $\phi \circ i = -\phi$.

Since Σ is stable, for any $\phi \in C^\infty(M)$, s.t. $\phi \circ i = \phi$, we have

$$\int_{\tilde{\Sigma}} |\nabla\phi|^2 \geq \int_{\tilde{\Sigma}} (Ric(\nu,\nu) + |A|^2)\phi^2 \tag{7.1}$$

Moreover, we have a conformal map $F : \tilde{\Sigma} \mapsto S^2(1) \hookrightarrow \mathbb{R}^3$, s.t. $degF \leq g(\Sigma) + 1, F \circ i = F$, where $g(\Sigma)$ is the genus of Σ. Assume $F = (f_1, f_2, f_3)$, then

$$\begin{aligned}
\int_{\tilde{\Sigma}} (Ric(\tilde{\nu},\tilde{\nu}) + |A|^2) &= \sum_{i=1}^{3} \int_{\tilde{\Sigma}} (Ric(\tilde{\nu},\tilde{\nu}) + |A|^2) f_i^2 \\
&\leq \sum \int_{\tilde{\Sigma}} |\nabla f_i|^2 = \int_{\Sigma} |\nabla F|^2 \\
&= 2deg(F)\text{Area}(S)(\text{integration by substitution}) \\
&= 8\pi deg(F) \leq 8\pi(g(\Sigma) + 1).
\end{aligned} \tag{7.2}$$

while

$$\begin{aligned}
\int_{\tilde{\Sigma}} Ric(\nu,\nu) + |A|^2 &= \frac{1}{2}\int_{\tilde{\Sigma}} R^M - R^\Sigma + |A|^2 \\
&\geq \frac{1}{2}\int_{\tilde{\Sigma}} 6 - K \\
&= \int_{\Sigma} 6 - K = 6\text{Area}(\Sigma) - 4\pi(1-g)
\end{aligned} \tag{7.3}$$

By (7.2) and (7.3), we have $\text{Area}(\Sigma) \leq 2\pi$, whenever $g = 0$ or 1.

(2)We then consider the 2-sided minimal surface $\Sigma \subset M$, with $ind(\Sigma) = 1$, i.e., there exists one and only one negative eigenvalue λ_1 for Jacobi operator L. Let u_1 be the eigenvector for λ_1, then for any $\phi \perp u_1$, we have

$$\int (Ric(\nu,\nu) + |A|^2)\phi^2 \leq \int |\nabla\phi|^2. \tag{7.4}$$

Since ϕ lies in a subspace that generated by nonnegative eigenvector. Moreover, there exist holomorphic map $F : \Sigma \mapsto S^2$, s.t. $degF \leq [\frac{3g+1}{4}]$, where [x] is the integer-valued function. Notice that

$$w = \int_{\Sigma} u_1 F \in B^3,$$

where $B_3 = \{x \in \mathbb{R}^3 : |x| < 1\}$. For any $v \in B^3$, there exist conformal function $F : B^3 \mapsto B^3$, s.t. $F_v(v) = 0$. Let $G = F_w \circ F$, we have

$$\int_\Sigma u_1 G = 0. \tag{7.5}$$

Let $G = (g_1, g_2, g_3)$, by (7.4) and (7.5), we have

$$\int L g_i g_i \geq 0.$$

Hence,

$$\int_\Sigma (Ric(\nu,\nu) + |A|^2 \leq \int_\Sigma |\nabla G|^2 \leq \deg G \mathrm{Area}(S^2) \leq 4\pi[\frac{3g+1}{4}]$$

while

$$\begin{aligned} \int_\Sigma Ric(\nu,\nu) + |A|^2 &= \frac{1}{2}\int_\Sigma R^M - R^\Sigma + |A|^2 \\ &\geq \frac{1}{2}\int_\Sigma 6 - K \\ &= \frac{1}{2}\int_\Sigma 6 - K = 3\mathrm{Area}(\Sigma) - 2\pi(1-g), \end{aligned} \tag{7.6}$$

we have $3\mathrm{Area}(\Sigma) \leq 2\pi(1-g) + 4\pi[\frac{3g+1}{4}]$.

Moreover, it's straightforward to check that $Ric(\nu,\nu) + |A|^2 \geq -2K$. So we have $4\pi(g-1) \leq 4\pi[\frac{3g+1}{4}]$. Consequently, $g \leq 3$.

- We have the following estimate:

Theorem 7.2 (Schoen-Yau). *Let $(\Sigma^2, \partial\Sigma) \hookrightarrow (M^3, g)$ be a stable minimal surface. Moreover, the scale curvature R_g of M has a positive lower bound Λ, i.e. $R_g \geq \Lambda > 0$. Then, we have $r_{\inf}(\Sigma, \partial\Sigma) \leq \frac{C}{\sqrt{\Lambda}}$ for some positive constant C, where $r_{\inf} = \max_{p\in\Sigma} dist(p, \partial\Sigma)$.*

Corollary 7.3. *Let (M, g) be a noncompact Riemannian manifold with $R_g \geq \Lambda > 0$, then there is no complete stable minimal surface in (M^3, g).*

Chapter 8

Criterion for stability

Theorem 8.1. *Let $\Sigma \subset M$ be a 2-sided minimal surface, Σ is stable iff*
(1)when Σ closed, there exist $u > 0$, s.t. $L_\Sigma u \leq 0$.
(2)when Σ is complete and noncomplete, there exist $u > 0$, s.t. $L_\Sigma u = 0$.

Remark 8.2. *This can be viewed as an infinitesimal version of the Calibration argument i.e. using foliation of minimal surfaces.*

Corollary 8.3. *$\Sigma \subset M$ is 2-sided minimal surface, Σ is stable, then for any covering map $\pi : \tilde{\Sigma} \mapsto \Sigma$, if $\tilde{\Sigma} \subset M$, then $\tilde{\Sigma}$ is also 2-sided and stable.*

Proof. Let $u > 0$ satisfy the condition in Theorem 8.1, then we have

$$L_{\tilde{\Sigma}}(u \circ \pi) = \begin{cases} \leq 0, \text{when } \Sigma \text{ is compact,} \\ = 0, \text{when } \Sigma \text{ is noncompact.} \end{cases}$$

□

Remark 8.4. *The "2-sided" condition is essential, consider $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$ is stable, but $S^2 \hookrightarrow \mathbb{R}P^3$ is not.*

Proof for Theorem 8.1. $\Longleftarrow$: Since $Lu = \Delta_\Sigma u + Qu \leq 0$, let $w = \log u (u > 0)$, we have

$$\Delta w = \frac{\Delta u}{u} - |\nabla w|^2 \leq -Q - |\nabla w|^2.$$

Then $\forall \varphi$ compactly supported,

$$\int_\Sigma \varphi^2 (\Delta w + Q) \leq - \int_\Sigma \varphi^2 |\nabla w|^2.$$

Using integration by part formula,

$$\int_\Sigma Q\varphi^2 \leq \int 2\varphi\langle\nabla\varphi,\nabla w\rangle - |\nabla w|^2\varphi^2 \leq \int 2|\varphi||\nabla\varphi||\nabla w| - |\nabla w|^2\varphi^2$$
$$\leq \int |\nabla\varphi|^2 + \varphi^2|\nabla w|^2 - |\nabla w|^2\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2.$$

Hence we have the stability inequality for Σ.
$\Longrightarrow$(1) Assume Σ is compact, then $\exists u > 0$, which is the first eigenfunction, such that $\lambda_1(\Sigma) \geq 0$, so

$$Lu = -\lambda_1 u \leq 0.$$

(2) Assume Σ is non-compact and stable, then Σ has an exhaustion $\Sigma = \cup_{i=1}^\infty \Omega_i$, and $\lambda_1(\Omega_i) > 0$ for all i. Now by elementary elliptic PDE, $\forall\psi \in C(\partial\Omega_i)$, $\exists! u$ in Ω_i, such that

$$\begin{cases} Lu_i = 0, \text{ in } \Omega_i, \\ u_i = \psi, \text{ on } \partial\Omega_i. \end{cases}$$

Claim: $\lambda_1(\Omega_i) > 0 \Longrightarrow$ if $\psi > 0$, then $u > 0$.
Otherwise, if $u \leq 0$, then $\Omega_{\{u\leq 0\}}$ has eigenvalue equals 0, since u is then a Dirichlet eigenfunction on $\Omega_{\{u\leq 0\}}$ with 0 boundary values, extend u by 0 to $\tilde{u} \in W^{1,2}(\Omega_i)$, let w be the $\lambda_1(\Omega_i)$ eigenfunction of L, then we have

$$0 < \lambda_1(\Omega) = \frac{\int_{\Omega_i}(-Lw)w}{\int_{\Omega_i}|w|^2} = \inf\{v \in W^{1,2}(\Omega_i) : \frac{\int_{\Omega_i}(-Lv)v}{\int_{\Omega_i}|v|^2}\} \leq \frac{\int_{\Omega_i}(-L\tilde{u})\tilde{u}}{\int_{\Omega_i}|\tilde{u}|^2} = 0, \tag{8.1}$$

which is a contradiction.

Now we can solve the boundary value problem for $u_i > 0$:

$$\begin{cases} Lu_i = 0 \text{ in } \Omega_i, \\ u = 1 \text{ on } \partial\Omega_i. \end{cases}$$

Fix $p \in \Omega_1$, then consider the normalized sequence $\{\frac{u_i}{u_i(p)}\}_{i=1}^\infty$,
Claim: Let $\tilde{u}_i = \frac{u_i}{u_i(p)}$, then there exists a subsequence $i' \to \infty, u \in C^\infty(\Sigma)$, such that

$$\tilde{u}_{i'} \to u \text{ in } C^2 \text{ on any compact subset of } \Sigma,$$

and

$$Lu = 0.$$

Proof of Claim: Fix any compact subset $K \subset \Sigma$, then when i big enough, $K \subset\subset \Omega_i$. Hence, we may assume $K \subset\subset \Omega_i$ for all $i > 0$. Fix Ω, Ω' s.t.

$K \subset\subset \Omega \subset\subset \Omega' \subset\subset \Omega_1$.
By Hanack inequality (cf. Theorem 4.3.3 [2]), we have

$$\sup_{x\in\Omega'} |\tilde{u}_i(x)| \leq C(n, \Omega', \Omega_1) \inf_{x\in\Omega_i} |\tilde{u}_i| \leq C(n, \Omega', \Omega_1)\tilde{u}(p) = C(n, \Omega', \Omega_1).$$

By a standard regularity theorem of EPDE, we could see that $\tilde{u}_i \in C^\infty(\Omega_1)$. Hence, by Gradient Estimate of EPDE (cf. Proposition 2.3.2 in [2]), we have

$$\sup_{x\in\Omega} |D\tilde{u}_i(x)| \leq C(n, \Omega, \Omega') \sup_{x\in\Omega'} |\tilde{u}_i(x)| \leq C(n, \Omega, \Omega', \Omega_1).$$

By Schauder Estimate, we have

$$\| \tilde{u}_i \|_{C^3(K)} \leq C(n, K, \Omega_1).$$

The claim follows from standard application of Ascoli.

The limit u is a positive solution of $Lu = 0$. □

Definition 8.5 (Convergence in the C^∞ sense)**.** *We say a sequence of minimal surface $\Sigma^{n-1} \subset \mathbb{R}^n$ converge to Σ in the sense of C^∞, if for any $p \in \Sigma$, there exist $r > 0$, s.t.*

$$\Sigma_i \cap B_r(p) = \textit{Graph of } u_i \textit{ over } T_p\Sigma \textit{ for } i \textit{ large enough,}$$

and $u_i \to u$ in C^∞ topology, where u is a function s.t. Σ is a graph of u over $T_p\Sigma$.

Example 8.6. *(1)Let $\Sigma \hookrightarrow M^n$ be the C^∞ embedding hypersurface. For any $p \in \Sigma$, $r_i \to 0$, $\Sigma_{p,r_i} = \frac{\Sigma - p}{r_i} \to T_p\Sigma$ in the sense of C^∞.*
(2)Let C be the Catenoid, then $\lambda_i C \to 2\mathbb{R}$ in C^∞ sense, except for 0.

Theorem 8.7 (L. Simon)**.** *If $\Sigma_i \to \Sigma$ in C^∞ sense, then there exists $u \neq 0$, s.t. $L_\Sigma u = 0$. If Σ lie in one side of Σ, we could find $u > 0$, s.t. $L_\Sigma u = 0$.*

Remark 8.8. *The Proof of Theorem 8.7 is similar to Theorem 8.1.*

Proposition 8.9. 1°. *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a 2-sided minimal surface, and if the Gauss image $G(\Sigma) \subset S_+^{n-1}$, then Σ is* ***stable****;*
2°. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a 2-sided minimal surface, and if $G(\Sigma) \subset U^{open} \subset S^2$, with $\mu_1(U) \geq 1$, where $\mu_1(U)$ is the Dirichlet eigenvalue of Δ_{S^2} on U, then Σ is stable. In perticular, $\mu_1(U) \geq 1$ is true if the area $|U| \leq 2\pi$.*

Proof. 1°. Let $e \in \mathbb{R}^n$ be the direction vector to the north pole, and let $u = e \cdot \nu$, where ν is the normal vector field of Σ, since the parallel translation in the e direction does not change the area of Σ, we have $\underline{Lu = 0}$. Since $G(\Sigma) \subset S_+^{n-1} \Longleftrightarrow e \cdot \nu > 0$, so $u > 0$, hence Σ is stable.
2°. $\mu_1(U) \geq 1 \Longrightarrow \exists v > 0$ on U such that

$$\begin{cases} \Delta_{S^2} v = \mu_i(U) v \leq -v, & \text{in } U, \\ v = 0, & \text{on } U. \end{cases}$$

Let $u = v \circ G$, where G is the Gauss Map. By Lemma 5.9, $G : \Sigma \to S^2$ is a conformal map, so

$$\Delta_\Sigma u = |A|^2 (\Delta_{S^2} v) \circ G \leq -|A|^2 u,$$

i.e. $Lu \leq 0$, hence Σ is stable. (In fact, on 2-dimension, the Jacobi operator $L = G^*(\Delta_{S^2} + 1)$.) $\square$

Chapter 9

The Generalized Bernstein Theorem

This chapter, we are going to prove:

Theorem 9.1 (The Generalized Bernstein Theorem)**.** *Any complete non-compact 2-sided stable minimal immersion* $\Sigma^2 \subset \mathbb{R}^3$ *is a plane.*

9.1 Bochner Technique

In this section, we will use Bochner Technique to prove:

Theorem 9.2 (Vanishing of Harmonic 1-Forms)**.** *If* $\Sigma^{n-1} \subset \mathbb{R}^n$ *is a complete, stable and 2-sided minimal surface, then any* L^2 *harmonic 1-form on* Σ *vanishes.*

Let (Σ^k, g) be a Riemannian manifold, and $\{e_1, \cdots, e_k\}$ an o.n. frame, with $\{\theta^1, \cdots, \theta^k\}$ the dual frame. Denote

$$(\nabla_{e_j}\alpha) = \sum_i \alpha_{i,j}\theta^i,$$

$$\nabla_{e_i}(\nabla\alpha) = \sum_{i,j} \alpha_{i,jk}\theta^i \otimes \theta^j;$$

then

$$\nabla\alpha = \sum_{i,j} \alpha_{i,j}\theta^i \otimes \theta^j, \quad \nabla^2\alpha = \sum_{i,j,k} \alpha_{i,jk}\theta^i \otimes \theta^j \otimes \theta^k.$$

Ricci Formula:

$$\boxed{\alpha_{i,jk} - \alpha_{i,kj} = \textstyle\sum_p \alpha_p R^{\Sigma}_{pijk}.}$$

Definition 9.3. *α is **harmonic** if $d\alpha = 0$ and $\delta\alpha = 0$(i.e. $\alpha_{i,j} = \alpha_{j,i}$ and $\sum_i \alpha_{i,i} = 0$).*

Bochner Formula: If α is harmonic, then

$$\Delta\alpha = Ric(\alpha^\sharp, \cdot),$$

where $\alpha^\sharp$ the vector field dual to α, and $\Delta\alpha = \sum_{i,j} \alpha_{i,jj}\theta^i$ is the rough laplacian.

Proof.

$$\sum_j \alpha_{i,jj} = \sum_j \alpha_{j,ij} = \underbrace{\sum_j \alpha_{j,ji}}_{=0} + \sum_{p,j} \alpha_p R^\Sigma_{pjij} = \sum_p \alpha_p Ric^\Sigma_{pi}.$$

□

Hence we have:

$$\boxed{\tfrac{1}{2}\Delta|\alpha|^2 = \langle \alpha, \Delta\alpha\rangle + |\nabla\alpha|^2 = Ric(\alpha^\sharp, \alpha^\sharp) + |\nabla\alpha|^2}.$$

In the case $\Sigma^{n-1} \subset \mathbb{R}^n$ is minimal, $R^\Sigma_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$ under the o.n. frame $\{e_i\}$ by the Gauss equation, hence

$$Ric^\Sigma_{ik} = \sum_j R^\Sigma_{ijkj} = -\sum_j h_{ij}h_{jk}, \quad (\sum_j h_{jj} = 0).$$

$$\Longrightarrow \quad \frac{1}{2}\Delta|\alpha|^2 = |\nabla\alpha|^2 + \sum_{ij} Ric^\Sigma_{ij}\alpha_i\alpha_j = |\nabla\alpha|^2 - \sum_i(\sum_j h_{ij}\alpha_j)^2 \geq |\nabla\alpha|^2 - |A|^2|\alpha|^2.$$

Plug in $\frac{1}{2}\Delta|\alpha|^2 = |\alpha|\Delta|\alpha| + \big|\nabla|\alpha|\big|^2$,

$$|\alpha|(\underbrace{\Delta|\alpha| + |A|^2|\alpha|}_{L|\alpha|}) \geq |\nabla\alpha|^2 - \big|\nabla|\alpha|\big|^2 \geq c(n)\big|\nabla|\alpha|\big|^2,$$

where $Lu = \Delta u + |A|^2 u$ is the stability operator, and $c(n)$ a constant depending only on n.

In general, choose the o.n. basis $\{e_1, \cdots, e_k\}$ such that under this basis $\alpha_1 = |\alpha|$ and $\alpha_j = 0$ for $j = 2, \cdots, k$, then

$$|\nabla\alpha|^2 - \big|\nabla|\alpha|\big|^2 = \sum_{ij} \alpha_{i,j}^2 - \frac{\sum_j(\sum_i \alpha_i\alpha_{i,j})^2}{|\alpha|^2} = \sum_{i,j} \alpha_{i,j}^2 - \sum_j \alpha_{1,j}^2$$

$$= \sum_{i>1,j} \alpha_{i,j}^2 \geq \sum_{i=2}^k \alpha_{i,i}^2 + \sum_{i=2}^k \alpha_{i,1}^2 \geq \frac{1}{k-1}(\underbrace{\sum_{i=2}^k \alpha_{i,i}}_{=-\alpha_{1,1}})^2 + \sum_{i=2}^k \alpha_{1,i}^2$$

$$\geq \frac{1}{k-1}[\alpha_{1,1}^2 + \sum_{i=2}^k \alpha_{1,i}^2] = \frac{1}{k-1}\left|\nabla|\alpha|\right|^2.$$

Proof of Theorem 9.2. 2-sided and stability means that $-\int_\Sigma \varphi L\varphi \geq 0$ for any φ compactly supported. So $\forall\varphi$ compactly supported

$$-\int_\Sigma \varphi|\alpha|L(\varphi|\alpha|) \geq 0,$$

i.e.

$$\int_\Sigma \varphi|\alpha|(\underbrace{\Delta(\varphi|\alpha|)}_{I} + |A|^2\varphi|\alpha|) \leq 0,$$

where

$$I = \int_\Sigma \varphi|\alpha|(\varphi\Delta|\alpha| + 2\langle\nabla\varphi, \nabla|\alpha|\rangle + |\alpha|\Delta\varphi) = \int_\Sigma \varphi^2|\alpha|\Delta|\alpha| + \frac{1}{2}\langle\nabla\varphi^2, \nabla|\alpha|^2\rangle + |\alpha|^2\varphi\Delta\varphi$$

$$\leq \int_\Sigma \varphi^2|\alpha|\Delta|\alpha| - \int_\Sigma \frac{1}{2}\Delta(\varphi^2)|\alpha|^2 + \int_\Sigma |\alpha|^2\varphi\Delta\varphi$$

$$= \int_\Sigma \varphi^2|\alpha|\Delta|\alpha| - (\varphi\Delta\varphi + |\nabla\varphi|^2)|\alpha|^2 + |\alpha|^2\varphi\Delta\varphi$$

$$= \int_\Sigma \varphi^2|\alpha|\Delta|\alpha| - |\nabla\varphi|^2|\alpha|^2.$$

Plug into the above

$$\int_\Sigma \varphi^2|\alpha|L(|\alpha|) \leq \int_\Sigma |\nabla\varphi|^2|\alpha|^2.$$

Now by taking $\varphi = \varphi_R$ to be cutoff functions on geodesic disk, and letting $R \to \infty$, the righthand side of the above inequality is zero, hence by $|\alpha|L(|\alpha|) \geq c(n)\left|\nabla|\alpha|\right|$ proved above,

$$c(n)\int_\Sigma \left|\nabla|\alpha|\right|^2 \leq \int_\Sigma |\alpha|L(|\alpha|) = 0,$$

which means that $|\alpha|$ is a constant, and hence is 0 since the area of Σ is ∞ by the monotonicity $|B_\sigma(p)| \geq w_k\sigma^k$. □

9.2 Proof of Generalized Bernstein Theorem

First, it's easy to see that:

Lemma 9.4. *Let (Σ^{2k}, g) be 2k-dimensional manifolds, for any $\omega \in \Omega^k\Sigma$, the L^2 norm of ω is comformally invariant.*

Lemma 9.5. *If α is harmonic in (Σ^2, g), then α is also harmonic in $(\Sigma, \rho g)$, for any smooth function $\rho > 0$.*

Proof. We notice that

$$\begin{cases} d\alpha = 0, \\ \delta_{\rho g}\alpha = \rho^{-2}\delta_g\alpha = 0. \end{cases}$$

□

Proposition 9.6. *$\tilde{\Sigma} \cong D^2$ cannot be conformally imersed into $\mathbb{R}^3$ as a complete noncompact 2-sided stable minimal surface.*

Proof. Let $(\mathbb{R}^3, \delta)$ be standard Euclidean space, and $i : \Sigma \mapsto D$ be the conformal map. By Lemma 9.4 and Lemma 9.5, we could see that L^2 harmonic 1-form on $(\tilde{\Sigma}, i * \delta)$ is one to one correspond to L^2 harmonic form on (D, δ).

Since there are many harmonic 1-forms on D by just taking df where f is harmonic functions, so it is a contradiction to the Theorem 9.2. □

Proof of Theorem 9.1. (Σ, g) is an oriented Riemann surface, where g is the restriction metric. If $z = x + iy$ then $g = \lambda^2(dx^2 + dy^2)$ locally. So Σ has a complex striation. Let $\hat{\Sigma}$ be the universal cover of Σ, then $\hat{\Sigma}$ is a simply connected non-compact Riemann surface, hence by uniformization theorem of Riemann surface,

$$\hat{\Sigma} \simeq \begin{cases} \mathbb{C}, & \text{the complex plane,} \\ D, & \text{the unit disk.} \end{cases}$$

By Proposition 9.6, second situation could not happen.

Hence, $\hat{\Sigma} \simeq \mathbb{C}$, then let $F : \mathbb{C} \to \mathbb{R}^3$, where $F = i \circ \pi$ is given by the composition of the minimal immersion $i : \Sigma \to \mathbb{R}^3$ with the covering map $\pi : \mathbb{C} \simeq \hat{\Sigma} \to \Sigma$. Since i is harmonic, and the harmonic?property is preserved under the conformal change $\hat{\Sigma} \simeq \mathbb{C}$, we know that F is both conformal and harmonic, i.e. $\Delta_{\mathbb{C}}F = 0$. Since Σ is stable and 2-sided, $\hat{\Sigma}$ is also stable and 2-sided, $\Longrightarrow \exists u > 0$, such that $Lu = \Delta_{\hat{\Sigma}}u + |\hat{A}|^2u = 0$ on $\hat{\Sigma}$. So $\Delta_{\hat{\Sigma}}u \leq 0$, hence $\Delta_{\mathbb{C}}(u \circ F) \leq 0$. So $u \circ F$ is a super-harmonic function. Since $\mathbb{C}$ has quadratic area growth,together with the fact that $u \circ F > 0$, we know that $u \circ F = 0$, and hence $|\hat{A}|^2 = 0$ by the following Proposition. □

Definition 9.7. *A Riemannian manifold* Σ^k *is called* ***parabolic*** *if every positive super-harmonic function is constant.*

Proposition 9.8. *If* $h : \Sigma \to \mathbb{R}^1_+$ *is a proper Lipschitz function* $|\nabla h| \leq c$*, and if* $|\Sigma_a| \leq ca^2$ *for some* $c > 0$*, where* $\Sigma_a = \{p \in \Sigma : \; h(p) \leq a\}$*, then* Σ *is parabolic.*

Proof. Take a positive super-harmonic function u, i.e. $\Delta u \leq 0$ and $u > 0$. Take $w = \log u$, then

$$\Delta w = \frac{\Delta u}{u} - |\nabla w|^2 \leq -|\nabla w|^2.$$

Take φ a compactly supported function,

$$\int_\Sigma \varphi^2 |\nabla w|^2 \leq -\int_\Sigma \Delta w \varphi^2 = \int 2\langle \nabla w, \varphi \rangle \varphi$$

$$\leq 2 \int |\varphi||\nabla w||\nabla \varphi| \leq \epsilon \int \varphi^2 |\nabla w|^2 + \frac{1}{\epsilon} \int |\nabla \varphi|^2.$$

Taking $\epsilon = \frac{1}{2}$, then

$$\int_\Sigma \varphi^2 |\nabla w|^2 \leq 4 \int_\Sigma |\nabla \varphi|^2.$$

By taking $h = dist_\Sigma(\cdot, p)$, we know that Σ has more than quadratic area growth, so we can take $\varphi = \varphi_R$ as in Lemma 5.13, and use the same logarithmic cut-off trick, to get $\int_\Sigma |\nabla \varphi_R|^2 \to 0$, and $\varphi_R \to 1$. So $|\nabla w| = 0$, and w hence u is a constant. □

Chapter 10

Curvature Estimate

Theorem 10.1. *Let $\Sigma_1^{n-1}, \Sigma_2^{n-1}$ be two connected, imbedded minimal hypersurfaces of $\mathbb{R}^n$, s.t.*

$$\Sigma_i \cap B_r(0) \neq \emptyset, B_r(0) \cap \partial\Sigma_i = \emptyset.$$

Moreover, we assume Σ_1 separate $B_r(0)$, and Σ_2 lie in one side of Σ_1, i.e.

$$B_r(0)\Sigma = U_1 \cup U_2, \Sigma_2 \cap B_r(0) \subset U_1,$$

where U_i is open and connected. Then, either $\Sigma_1 \cap \Sigma_2 = \emptyset$ or $\Sigma_1 = \Sigma_2$.

Proof. Assume $p \in \Sigma_1 \cap \Sigma_2$, then locally, they can be written as graphs of u^1, u^2, s.t.

$$(\delta_{ij} - \frac{u_i^\alpha u_j^\alpha}{1 + |\nabla u^\alpha|})u_{ij}^\alpha = 0, \alpha = 1, 2.$$

Substract the two equations, we have

$$\delta_{ij}(u_{ij}^1 - u_{ij}^2) + b_i(u^1 - u^2)_i + c(u^1 - u^2) = 0,$$

for some b_i, c. Hence by maximal principle, $u^1 = u^2$ locally. $\square$

Let $\Sigma^{n-1} \hookrightarrow \mathbb{R}^n$ be a immersed minimal hypersurface, $0 \in \Sigma$, let

$$L_r = \{\Sigma \cap B_r(0) \neq \emptyset, \partial\Sigma \cap B_r(0) = \emptyset\},$$

we want to prove

$$|A|^2(x) \leq \frac{C}{d^2(x, \partial B_r(0))}.$$

By a suitable dilation and translation, we just need to prove

$$|A|^2(0) \leq C.$$

Let $\{e_1, \cdots, e_{n-1}\}$ be local o.n. frames on Σ, and denote $h_{ij,klm}$ by the covariant derivatives of the second fundamental form h on Σ. The rough laplacian for h is defined as

$$\Delta h_{ij} = \sum_{k=1}^{n-1} h_{ij,kk}.$$

Proposition 10.2.

$$\Delta h_{ij} + |A|^2 h_{ij} = 0, \quad 0 \leq i, j \leq n-1 \tag{10.1}$$

Proof. Firstly we have the Ricci identity:

$$h_{ij,kl} - h_{ij,lk} = \sum_p h_{pj} R_{pikl} + \sum_p h_{ip} R_{pjkl},$$

Gauss Equation:

$$R^{\Sigma}_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

and Codazzi equation:

$$h_{ij,k} = h_{ik,j}.$$

Using the Eistein summation, we have

$$\begin{aligned}\Delta h_{ij} = h_{ij,kk} = h_{ik,jk} = &\underbrace{h_{ik,kj}}_{=h_{kk,ij}=0} + h_{pk}R^{\Sigma}_{pijk} + h_{ip}R^{\Sigma}_{pkjk} \\ &= h_{pk}(h_{pj}h_{ik} - h_{pk}h_{ij}) + h_{ip}(h_{pj}\underbrace{h_{kk}}_{=0} - h_{pk}h_{kj}) \\ &= -|A|^2 h_{ij} + (\underbrace{h_{ik}h_{kp}h_{pj} - h_{ip}H_{pk}h_{kj}}_{=0}).\end{aligned}$$

So we finished the proof. □

Now recall that the stability operator is $L\varphi = \Delta\varphi + |A|^2\varphi$.

Proposition 10.3.

$$|A|(L|A|) \geq \frac{2}{n-1}\big|\nabla|A|\big|^2. \tag{10.2}$$

Proof. By the Bochner Formula,

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \langle A, \Delta A\rangle = |\nabla A|^2 - |A|^4.$$

While $\frac{1}{2}\Delta|A|^2 = |A|\Delta|A| + |\nabla|A||^2$,

$$|A|L(|A|) = |\nabla A|^2 - \big|\nabla|A|\big|^2 = \sum_{i,j,k} h_{ij,k}^2 - \frac{\sum_k(\sum_{ij} h_{ij}h_{ij,k})^2}{|A|^2}. \tag{10.3}$$

In an o.n. eigenbasis $\{e_1, \cdots, e_{n-1}\}$ of h, $h_{ij} = \lambda_i\delta_{ij}$, so

$$\big|\nabla|A|\big|^2 = \frac{\sum_k(\sum_i \lambda_i h_{ii,k})^2}{|A|^2} \leq \sum_{i,k} h_{ii,k}^2 = \sum_{i\neq k} h_{ii,k}^2 + \sum_i h_{ii,i}^2$$

$$= \sum_{i\neq k} h_{ii,k}^2 + \sum_i(-\sum_{j\neq i} h_{jj,i})^2 \leq \sum_{i\neq k} h_{ii,k}^2 + (n-2)\sum_{i\neq j} h_{jj,i}^2$$

$$= (n-1)\sum_{i\neq k} h_{ii,k}^2 = \frac{n-1}{2}(\sum_{i\neq k} h_{ik,i}^2 + \sum_{i\neq k} h_{ki,i}).$$

So

$$(1 + \frac{2}{n-1})\big|\nabla|A|\big|^2 \leq \sum_{i,k} h_{ii,k}^2 + \sum_{i\neq k} h_{ik,i}^2 + \sum_{i\neq k} h_{ki,i}^2 \leq \sum_{i,j,k} h_{ij,k}^2 = |\nabla A|^2.$$

So we finished the proof. $\square$

Chapter 11

Choi-Schoen's Theorem

Theorem 11.1 (Choi-Schoen [4]). *Suppose $\Sigma^2 \subset M^3$ is a minimal surface. Assume $0 \in \Sigma^2$, and $\partial\Sigma \cap B_{r_0}(0) = \emptyset$. Moreover, there exists $\epsilon, \rho > 0$ (depending only on M), such that $\int_{\Sigma \cap B_{r_0}} |A|^2 < \epsilon$, then*

$$d^2(x, \partial B_\rho(0))|A|^2(x) \leq \delta.$$

Proof. Let us give a proof when $M^3 = \mathbb{R}^3$, and the general cases follow by the fact that M^3 is locally near $\mathbb{R}^3$ when ρ is small enough. Assume $\delta = 1$ and

$$F(x) = d^2(x, \partial B_\rho(0))|A|^2(x),$$

Since $F|_{\partial B_{r_0}} = 0$, then $\exists x_0 \in B_\rho$, such that $F(x_0) = \max_{B_{r_0}} F(x)$.
Need to show: $F(x_0) \leq 1$.
Suppose otherwise $F(x_0) > 1$, let $\delta = \frac{\rho - r(y_0)}{2}$, then

- $\sup_{B_\delta(y)} |A|^2 \leq 4|A|^2(x_0)$.
  This is because $d^2(x, \partial B_\rho(0))|A|^2(x) \leq 4d^2(x_0, \partial B_\rho(0))|A|^2(x_0)$, hence $|A|^2(x) \leq (\frac{d^2(x_0,\partial B_\rho(0))}{d^2(x,\partial B_\rho(0))})^2|A|^2(x_0) \leq (\frac{d^2(x_0,\partial B_\delta(x_0))}{d^2(x,\partial B_\delta(x_0))})^2|A|^2(x_0) \leq 4|A|^2(x_0)$.
- $(2\delta)^2|A|^2(x_0) = F(x_0) > 1 \Longrightarrow \delta^2|A|^2(x_0) > 1/4$.

Let $\delta_0 = \frac{1}{|A|(x_0)}$, hence $\delta^2 \geq \frac{1}{4}\delta_0^2 \Longrightarrow \delta_0/2 < \delta$. So $B_{\delta_0/2}(x_0) \subset B_\delta(x_0)$. Let

$$\Sigma_{\delta_0} = \frac{2}{\delta_0}(\Sigma - x_0),$$

$$\Longrightarrow \begin{cases} \sup_{B_1} |A_{\Sigma_{\delta_0}}|^2 = 4|A_{\Sigma_{\delta_0}}|^2 = \delta_0^2|A|^2(x_0) = 1, \\ \int_{B_1} |A_{\Sigma_{\delta_0}}|^2 \leq \epsilon. \end{cases}$$

By Simon's equation, let $A = A_{\Sigma_{\delta_0}}$, on B_1, we have

$$\Delta|A|^2 = 2(|\nabla A|^2 - |A|^4) \geq -2|A|^4 \geq -2|A|^2$$

Hence by mean value property of subsolution of EPDE, we have $|A(0)|^2 \leq c\int_{\Sigma\cap B_1} |A|^2 \leq c\epsilon$, which is a contradiction. □

Corollary 11.2. *Assume $\Sigma^2 \subset \mathbb{R}^3$ is stable and 2-sided with quadratic area growth, i.e.* $\text{Area}(\Sigma \cap B_{r_0}) \leq cr_0^2$*, then*

$$\sup_{\Sigma\cap B_{r_0/2}} |A|^2 \leq cr_0^{-2}.$$

Proof. By stability, we have $\int_\Sigma |A|^2\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2$. Since Σ has quadratic area growth, we can use the logarithmic cutoff trick to get,

$$\int_{\Sigma\cap B_{r_0/k}} |A|^2 \leq \frac{C}{\log k}, \quad k \gg 1.$$

So for k large enough, we have $\int_{\Sigma\cap B_{r_1}(y)} |A|^2 < \epsilon$, where $r_1 = r_0/k$, hence

$$|A|^2(y) \leq cr_1^{-2} \leq c'r_0^{-2}, \quad c' = kc.$$

□

Lemma 11.3 (Generalized Monotone formula)**.** *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal surface, $\partial\Sigma \cap B_\rho(0) = \emptyset$, $f : \Sigma \mapsto \mathbb{R}$ is smooth, then*

$$\frac{\int_{\Sigma\cap B_t} f dvol}{t^k} - \frac{\int_{\Sigma\cap B_s} f dvol}{s^k}$$

$$= \int_{\Sigma\cap(B_t\setminus B_s)} f\frac{|x^\perp|^2}{|x|^{k+2}} dvol + \frac{1}{2}\int_s^t \frac{1}{\tau^{k+1}}\int_{\Sigma\cap B_\tau} (\tau^2 - |x|^2)\Delta_\Sigma f dvol d\tau,$$

where $0 < s < t$, and $x^\perp$ is the projection of x to the normal part of $T_x\Sigma$.

Proof. First, we have

$$\int_{\Sigma\cap B_t} \text{div}(fx) dvol_\Sigma = \int_{\partial(\Sigma\cap B_t)} fx\cdot\mathbf{n} d\sigma = \int_{\partial(\Sigma\cap B_t)} f|x^T| d\sigma, \tag{11.1}$$

where $\mathbf{n} = \frac{x^T}{|x^T|}$ is exterior normal vector field on $\partial(\Sigma \cap B_t)$. While

$$\begin{aligned}\int_{\Sigma\cap B_t} \text{div}(fx) dvol_\Sigma &= \int_{\Sigma\cap B_t} \nabla f\cdot x dvol_\Sigma + \int_{\Sigma\cap B_t} f\text{div}(x) dvol_\Sigma \\ &= \int_{\Sigma\cap B_t} \nabla f\cdot x dvol_\Sigma + \int_{\Sigma\cap B_t} kf dvol_\Sigma = I + \int_{\Sigma\cap B_t} kf dvol_\Sigma.\end{aligned} \tag{11.2}$$

We have estimate

$$
\begin{aligned}
I &= \int_{\Sigma\cap B_t} \nabla f \nabla(\frac{|x|^2}{2}) dvol_\Sigma \\
&= \int_{\Sigma\cap B_t} \mathrm{div}(\frac{|x|^2}{2} f) - \frac{|x|^2}{2}\Delta_\Sigma f dvol_\Sigma \\
&= \int_{\partial(\Sigma\cap B_t)} \frac{t^2}{2}\nabla f\cdot \mathbf{n} d\sigma - \int_{\Sigma\cap B_t}\frac{|x|^2}{2}\Delta_\Sigma f dvol_\Sigma \\
&= \int_{\Sigma\cap B_t} (\frac{t^2-|x|^2}{2})\Delta_\Sigma f dvol_\Sigma
\end{aligned}
\tag{11.3}
$$

Take $h(x) = |x|$, then $\{h \le t\} = B_t(0)$, then $\nabla h = \frac{x^T}{|x|}$ For the right hand side of (11.1), we have

$$
\begin{aligned}
\int_{\partial(\Sigma\cap B_t)} f|x^T| d\sigma &= t\int_{\partial(\Sigma\cap B_t)} f\frac{|x^T|}{|x|} d\sigma \\
&= t\frac{d}{dt}\int_{\Sigma\cap B_t} f\frac{|x^T|}{|x|}|\nabla h| dvol \\
&= t\frac{d}{dt}\int_{\Sigma\cap B_t} f\frac{|x^T|^2}{|x|^2} \\
&= t\frac{d}{dt}\int_{\Sigma\cap B_t} f(1-\frac{|x^\perp|^2}{|x|^2}) \\
&= t\frac{d}{dt}\int_{\Sigma\cap B_t} f - \int_{\partial(\Sigma\cap B_t)} f\frac{|x^\perp|^2}{|x^T|} (\text{ By coarea formula})
\end{aligned}
\tag{11.4}
$$

Let $F(t) = \int_{\Sigma\cap B_t} f$, by (11.1),(11.2),(11.3) and (11.4), we have

$$
kF(t) = tF'(t) - \int_{\partial(\Sigma\cap B_t)} f\frac{|x^\perp|^2}{|x^T|} - \int_{\Sigma\cap B_t}(\frac{t^2-|x|^2}{2})\Delta_\Sigma f dvol_\Sigma. \tag{11.5}
$$

The lemma then follows easily from the above ODE. □

Lemma 11.4 (Mean Value Properity)**.** *If $\Delta_\Sigma f \ge -cf$ in $\Sigma\cap B_1$, $f>0$, then*

$$
f(0) \le c'\int_{\Sigma\cap B_1} f dvol_\Sigma.
$$

Proof. Since $\Delta_\Sigma f \geq -cf$, (11.5) imply that

$$\begin{aligned}\frac{d}{dt}(t^{-k}F(t)) &= t^{-k-1}(\int_{\partial(\Sigma\cap B_t)} f\frac{|x^\perp|^2}{|x^T|} + \int_{\Sigma\cap B_t}(\frac{t^2-|x|^2}{2})\Delta_\Sigma f dvol_\Sigma) \\ &\geq -\frac{c}{2}t^{-k+1}\int_{\Sigma\cap B_t} f dvol_\Sigma),\end{aligned}$$

where $F(t) = \int_{\Sigma\cap B_t} f$. Hence, we know that $e^{ct/2}F(t)$ is increasing. Hence $f(0) = F(0) \leq e^{c/2}F(1) = e^{c/2}\int_{\Sigma\cap B_1} f$. □

Chapter 12

Small Curvature implies Graphical

Let us firstly give a technical lemma used in the argument of the above section.

Lemma 12.1. *$\Sigma^2 \subset \mathbb{R}^n$ is minimal. Assume that $s^2 \sup_\Sigma |A|^2 \le \frac{1}{16}$. If $x_0 \in \Sigma^2$ and $dist_\Sigma(x_0, \partial\Sigma) \ge 2s$ (i.e. $\partial\Sigma \cap B_{2s}(x) = \emptyset$), then*

(i) $B^\Sigma_{2s}(x_0)$ is graphical over $T_{x_0}\Sigma$ of some function u, where $B^\Sigma_{2s}(x_0)$ is the geodesic ball of Σ, and $|\nabla u| \le 1$ and $|Hessu| \le \frac{1}{\sqrt{2}s}$;

(ii) Let Σ' be a connected component of $B_s(x) \cap \Sigma$ containing x_0, then $\Sigma' \subset B^\Sigma_{2s}(x_0)$.

Proof. Define $d(x,y) = dist_{S^{n-1}}(N(x), N(y))$. Connect x to $y \in \Sigma \cap B^\Sigma_{2s}(x_0)$ by unit speed geodesic $u : [0,r] \mapsto \Sigma$, s.t. $u(0) = x_0, u(r) = y$. Then

$$d(x_0, y) \le \int_0^r |\nabla^\Sigma N| dt \le \int_0^r |A| dt \le \int_0^r \frac{1}{4s} dt < \frac{1}{2} < \frac{\pi}{4},$$

Hence $\Sigma \cap B^\Sigma_{2s}(x_0)$ is contained in the graph of a function u over a subset of $T_{x_0}\Sigma$. Since

$$\frac{1}{\sqrt{2}} \ge cos(d(x,y)) =< N(x_0), N(y) >= \frac{1}{\sqrt{1 + |\nabla u(y)|^2}},$$

we have $|\nabla u(y)| \le 1$. Since

$$A = \frac{1}{1+|\nabla u|^2} \begin{pmatrix} u_{x_1x_1} & \dots & u_{x_1x_{n-1}} \\ \vdots & \ddots & \vdots \\ u_{x_{n-1}x_1} & \dots & u_{x_{n-1}x_{n-1}} \end{pmatrix}, g = \begin{pmatrix} 1+u^2_{x_1} & \dots & u_{x_1}u_{x_{n-1}} \\ \vdots & \ddots & \vdots \\ u_{x_{n-1}}u_{x_1} & \dots & 1+u^2_{x_{n-1}} \end{pmatrix}, \tag{12.1}$$

and

$$|A|^2 = A_{ik}A_{jl}g^{ij}g^{kl},$$

we have

$$\frac{|Hess(u)|^2}{(1+|\nabla u|^2)^3} \leq |A|^2 \leq \frac{1}{16s^2},$$

Consequently,

$$|Hess(u)| \leq \frac{1}{\sqrt{2}s}$$

Now we are going to prove:
If $y \in B^{\Sigma}_{2s}(x_0)$, then $d^{\mathbb{R}^n}(y, x_0) > s$. Therefore, $\Sigma \cap B^{\mathbb{R}^n}_s(x_0) \subset B^{\Sigma}_{2s}(x_0)$. Let $w : [0, 2s] \mapsto \Sigma$ be a unital speed geodesic, and $w(0) = x_0, w(2s) = y$. Since

$$|w(2s - w(0))| = |w(2s - u(0))||w'(0)| \geq \langle w(2s) - u(0), w'(0)\rangle,$$

in order to prove $|w(2s) - w(0))| > s$, we just need to show

$$\langle w(2s) - w(0), w'(0)\rangle > s.$$

Let $f(t) = \langle w(t) - w(0), w'(0)\rangle$, we have $f(0) = 0, f'(0) = 1$. Moreover

$$|f''(t)| = |\langle u''(t), u(0)\rangle| < \frac{1}{4s},$$

which mean

$$f(2s) = f(0) + 2f'(0)s + \frac{1}{2}f''(\xi)(2s)^2 > 2s - \frac{1}{4s}(2s)^2 = s.$$

□

Remark 12.2. *Let $U \subset \mathbb{R}^3$, then $L_{c_1,c_2} = \{\Sigma \subset U, \partial\Sigma \cap U = \emptyset, H(\Sigma) \equiv 0, \max_U |A^{\Sigma}| \leq c_1, \mathrm{Area}(\Sigma) \leq c_2\}$ is closed in C^∞ topology.*

Theorem 12.3. *Let $\Sigma^k \subset \mathbb{R}^n$ be minimal. $\exists \epsilon = \epsilon(n, k)$, if $x \in \Sigma$, $\partial\Sigma \subset \partial B_{r_0}(x)$, and $\Theta_x(r_0) - 1 < \epsilon$, then*

$$\sup_{\Sigma \cap B_{r_0/2}(x)} |A|^2 \leq r_0^{-2}.$$

Proof.

- It suffices to assume that $\Theta_y(r_1) - 1 < \epsilon$ for all $y \in B_{r_1}(x) \cap \Sigma$ by the monotonicity formula 5.1.

- By rescaling the function $(r_1 - |y|)^2|A|^2(y)$ near the maximum point as in the proof of Theorem 11.2, we can get another minimal surface, denoted still as Σ, such that $0 \in \Sigma$, $\partial\Sigma \subset \partial B_1(0)$, $|A|^2 \leq 1$ on Σ, and $|A|^2(0) = \frac{1}{4}$. Furthermore, by the small excess condition, $|\Sigma \cap B_1(0)| \leq (1+\epsilon)\omega_k$.
- This is not possible if $\epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ small enough, by the following argument.
- Compactness argument: consider the class

  $$\mathcal{C}_\epsilon = \{\Sigma :\ 0 \in \Sigma,\ \partial\Sigma \subset \partial B_1(0),\ |A|^2 \leq 1,\ |A|^2(0) = \frac{1}{4},\ |\Sigma \cap B_1(0)| \leq (1+\epsilon)\omega_k\}.$$

  If the curvature estimates is not true, then we can find a sequence $\{\Sigma_i\}$, with $\Sigma \cap B_1(0) \leq (1+2^{-i})\omega_k$. A subsequence $\Sigma_i \to \Sigma$ in C^k norm to some minimal Σ_∞, such that $\Sigma_\infty \in \mathcal{C}_0$, i.e. $|\Sigma_\infty \cap B_1(0)| = \omega_k$, $\Longrightarrow \Sigma$ is a disk, hence contradiction to the curvature assumption $|A|^2(0) = \frac{1}{4}$.

$\square$

Chapter 13

Schoen's Curvature Estimate

Theorem 13.1. *Assume that Σ^2 is stable and 2-sided in $\mathbb{R}^3$. If $x \in \Sigma$ and $dist(x, \partial\Sigma) \geq r_0$, then*

$$\text{Area}(B^{\Sigma}_{r_0}(x_0) \cap \Sigma) \leq \frac{4\pi}{3} r_0^2.$$

Proof. It suffices to assume $\pi_1(\Sigma) = \{1\}$, or we can pass to the universal cover $\tilde{\Sigma}$ of Σ, which is also stable and 2-sided. Since $\text{Area}(B^{\tilde{\Sigma}}_{r_0}(x_0) \cap \Sigma) \geq \text{Area}(B^{\Sigma}_{r_0}(x_0) \cap \Sigma)$, we can get the result. Let $\phi(x) = r_0 - r(x, x_0)$, then $\phi \in C^1_c(\Sigma \cap B^{\Sigma}_{r_0}(x_0))$ and $|\nabla \phi| = 1$ a.e.

Moreover, by Stability, we have

$$\int_{\Sigma \cap B^{\Sigma}_{r_0}(x_0)} |A|^2 \phi^2 dvol_{\Sigma} \leq \int_{\Sigma \cap B^{\Sigma}_{r_0}(x_0)} |\nabla \phi|^2 dvol_{\Sigma} = \text{Area}(\Sigma \cap B^{\Sigma}_{r_0}(x_0)). \tag{13.1}$$

While

$$\int_{\Sigma \cap B^{\Sigma}_{r_0}(x_0)} |A|^2 \phi^2 dvol_{\Sigma} = \int_0^{r_0} \int_{\partial(\Sigma \cap B^{\Sigma}_{r}(x_0))} |A|^2 d\sigma |r_0 - r|^2 dr \tag{13.2}$$

Let $f(r) = \int_0^r \int_0^s \int_{\partial(\Sigma \cap B^{\Sigma}_{\tau}(x_0))} |A|^2 d\sigma d\tau ds = \int_0^r \int_{\Sigma \cap B^{\Sigma}_{s}(x_0)} |A|^2 dvol_{\Sigma} ds$, $g(r) = |r - r_0|^2$. Then we have

$$f(0) = g(0) = 0, f'(0) = g'(0) = 0.$$

Moreover,

$$\begin{aligned} \text{LHS of (13.2)} &= \int_0^{r_0} f''(r) g(r) dr = \int_0^{r_0} f(r) g''(r) \\ &= 2 \int_0^{r_0} \int_0^r \int_{\Sigma \cap B^{\Sigma}_{\tau}(x_0)} |A|^2 dvol_{\Sigma} d\tau dr. \end{aligned} \tag{13.3}$$

Since

$$\frac{d}{dr}Length(\Sigma\cap\partial B_r^\Sigma) = \int_{\Sigma\cap\partial B_r^\Sigma(x_0)} k_g d\Sigma = 2\pi - \int_{\Sigma\cap B_r^\Sigma} K dvol_\Sigma$$

$$= 2\pi + \frac{1}{2}\int_{\Sigma\cap B_r^\Sigma(x_0)} |A|^2 dvol_\Sigma,$$

we have

$$\text{Area}(\Sigma\cap B_{r_0}^\Sigma) = \int_0^{r_0} Length(\Sigma\cap\partial B_r^\Sigma)dr = \pi r_0^2 + \frac{1}{2}\int_0^{r_0}\int_{\Sigma\cap B_r^\Sigma(x_0)} |A|^2 dvol_\Sigma dr.$$

Combine with (13.1), (13.2) and (13.3), we have

$$\text{Area}(\Sigma\cap B_{r_0}^\Sigma) < \frac{4}{3}\pi r_0^2.$$

□

Hence, by Corollary 11.2, we have

Corollary 13.2. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a minimal surface, $x_0 \in \Sigma$, and $\partial\Sigma \cap B_{r_0}^{\mathbb{R}^3}(x_0) = \emptyset$. If Σ is stable and two-sided, then*

$$\sup_{\Sigma\cap B_{r_0}^{\mathbb{R}^3}(x_0)} |A|^2 \le C/r_0^2.$$

Theorem 13.3. *If $p < 2 + \sqrt{\frac{2}{n-1}}$, then $\forall \phi \in C_c^1(B_{r_0}(x_0)\cap\Sigma)$, we have*

$$\int_{B_{r_0}(x_0)\cap\Sigma} |A|^{2p}\phi^{2p} \le C(p)\int_\Sigma |\nabla\phi|^{2p}.$$

Proof. First, we claim:

$$\frac{2}{n-1}\int_\Sigma \big|\nabla|A|\big|^2\varphi^2 \le \int_\Sigma |\nabla\varphi|^2|A|^2, \quad \forall\varphi\in C_c^1(\Sigma).$$

By plug in $\varphi|A|$ to the stability inequality $-\int_\Sigma(\varphi|A|)L(\varphi|A|) \ge 0$, and using the tricks in Theorem 9.2, we have

$$\int_\Sigma \varphi^2|A|L(|A|) \le \int_\Sigma |\nabla\varphi|^2|A|^2.$$

Using Proposition 10.3, we can prove the claim.

Now change $\varphi \to \varphi|A|^q$, for some $q > 0$, then we get

$$\frac{2}{n-1}\int_\Sigma \big|\nabla|A|\big|^2\varphi^2|A|^{2q} \le \int_\Sigma |A|^2\big|\nabla(\varphi|A|^q)\big|^2 = \int_\Sigma |A|^2\big|(\nabla\varphi)|A|^q + q|A|^{q-1}\varphi(\nabla|A|)\big|^2$$

$$\le (q^2+\epsilon)\int_\Sigma |A|^{2q}\varphi^2\big|\nabla|A|\big|^2 + (1+\frac{1}{\epsilon})\int_\Sigma |\nabla\varphi|^2|A|^{2q+2}.$$

Hence if $\boxed{q < \sqrt{\frac{2}{n-1}}}$, by moving the first term on the right hand side to the left,

$$\Longrightarrow \int_\Sigma \big|\nabla|A|\big|^2\varphi^2|A|^{2q} \le C(q)\int_\Sigma |A|^{2q+2}|\nabla\varphi|^2,$$

$$\Longrightarrow \int_\Sigma \big|\nabla|A|^{q+1}\big|\varphi^2 \le C(q)\int_\Sigma (|A|^{q+1})^2|\nabla\varphi|^2.$$

Set $p = q+2$,

$$\Longrightarrow \int_\Sigma \big|\nabla|A|^{p-1}\big|\varphi^2 \le C(p)\int_\Sigma (|A|^{p-1})^2|\nabla\varphi|^2.$$

Now replace φ by $\varphi|A|^{p-1}$ in the stability inequality $\int_\Sigma |A|^2\varphi^2 \le \int_\Sigma |\nabla\varphi|^2$,

$$\Longrightarrow \int_\Sigma |A|^{2p}\varphi^2 \le \int_\Sigma \big|\nabla(\varphi|A|^{p-1})\big|^2 \le 2\int_\Sigma \varphi^2\big|\nabla(|A|^{p-1})\big|^2 + |\nabla\varphi|^2|A|^{2p-2}.$$

Using the above inequality,

$$\Longrightarrow \quad \int_\Sigma |A|^{2p}\varphi^2 \le C(p)\int_\Sigma |A|^{2p-2}|\nabla\varphi|^2.$$

Replace φ by φ^p, then

$$\int_\Sigma |A|^{2p}\varphi^2 \le C(p)\int_\Sigma |A|^{2p-2}|\nabla\varphi^p|^2 = Cp^2\int_\Sigma (\varphi|A|)^{2p-2}|\nabla\varphi|^2$$

$$\le C(p)\Big\{\int_\Sigma (\varphi|A|)^{2p}\Big\}^{(p-1)/p}\Big\{\int_\Sigma |\nabla\varphi|^{2p}\Big\}^{1/p}.$$

$$\Longrightarrow \quad \underline{\int_\Sigma (\varphi|A|)^{2p} \le C(p)\int_\Sigma |\nabla\varphi|^{2p}}, \quad \forall p < 2 + \sqrt{\frac{2}{n-1}}. \qquad (13.4)$$

□

Theorem 13.4. *(Schoen-Simon-Yau [5]) Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a stable 2-sided minimal surface. Assume $x_0 \in \Sigma$, $\partial\Sigma \subset \partial B_{r_0}(x_0)$, $|\Sigma \cap B_{r_0}(x_0)| \leq V r_0^{n-1}$ and $n \leq 6$. Then*

$$\sup_{\Sigma \cap B_{r_0/2}(x_0)} |A|^2 \leq c(n, V) r_0^{-2}.$$

Proof. When $n \leq 6$, take $2p = n - 1$, and use the logarithmic cut off trick and the volume growth $\Longrightarrow \int_\Sigma |A|^{n-1}$ is small in small ball, and hence the curvature estimates. □

Corollary 13.5. *A complete 2-sided stable $\Sigma^{n-1} \in \mathbb{R}^n$ with $\mathbb{R}^{n-1}$ volume growth and $n \leq 6$ is a hyperplane.*

Proof. Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be complete, stable, 2-sided, $n \leq 6$ and $\Sigma \cap B_R \leq CR^{n-1} \Longrightarrow \Sigma$ is hyperplane.

Take $2p > n - 1$, $\Longrightarrow \int_\Sigma (\varphi|A|)^{2p} \leq C \int_\Sigma |\nabla\varphi|^{2p} \leq \frac{C}{R^{2p}} |\Sigma \cap B_{2R}| \to 0$. □

Chapter 14

Bernstein's Theorem and Minimal Cone

Theorem 14.1. *If $u : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ is an entire solution if the minimal surface equation, and $n \leq 8$, then u is linear.*

Proof. Let Σ_u be the minimal graph of u, and assume $0 \in \Sigma_u$. Recall monotonicity formula:

$$\frac{\mathrm{Area}(\Sigma_u \cap B_R)}{R^{n-1}} - \frac{\mathrm{Area}(\Sigma_u \cap B_r)}{r^{n-1}} = \int_{\Sigma \cap (B_R \setminus B_r)} \frac{|x^\perp|^2}{|x|^{n+1}} d\sigma. \tag{14.1}$$

Consider the density at infinite $\Theta_\infty := \lim_{r\to\infty} \frac{\mathrm{Area}(\Sigma_u \cap B_r)}{w^{n-1} r^{n-1}}$. By area bound of area-minimizing surface, we have

$$\Theta_\infty < \frac{\mathrm{Area}\partial B_1^n}{2\mathrm{Area}B_1^{n-1}}.$$

We need the following lemma to complete the proof:

Lemma 14.2. *We have $\Theta_\infty \geq 1$ and the equality holds iff Σ is a plane.*

Proof. It's clear that $\Theta_\infty \geq 1$. When $\Theta_\infty = 1$, since we know that the density of Σ is 1 at original point, in (14.1), let $R \to \infty$, $r \to 0$, we have $x^\perp \equiv 0$, i.e. $x \in T_x\Sigma$.

We then claim: If $x \in T_x\Sigma$, Σ is a cone.

This is because, consider the following linear equations in $\mathbb{R}^n$,

$$\begin{cases} \frac{d}{dt}x(t) = x(t), \\ x(0) = x_0, \end{cases}$$

Then we can see that $x(t) = e^t x_0$, which is a straight line pass x_0. Since $x \in T_x\Sigma$, we could see that $x(t) \in \Sigma, \forall t$. Hence, Σ is a cone.

Then blow up at original, since Σ_u is a cone, we have

$$\Sigma_u = \lim_{n\to\infty} \frac{\Sigma_u}{r_n} = T_0\Sigma_u,$$

where $r_n \to 0$. □

By the lemma, if Σ_u is not affine, then we have $\Theta_\infty > 1$. We then consider blow-down of Σ_u at 0, i.e. consider

$$\Sigma_\infty = \lim_{n\to\infty} r_n\Sigma_u,$$

with $r_n \to 0$.

We claim that

$$\frac{\text{Area}\Sigma_\infty \cap B_{\mathbb{R}^n}}{w_{n-1}R^{n-1}} \equiv \Theta_\infty.$$

This is simply because,

$$\frac{\text{Area}(\Sigma_\infty \cap B_R^{\mathbb{R}^n})}{w_{n-1}R^{n-1}} = \lim_{n\to\infty} \frac{\text{Area}(r_n\Sigma_u \cap B_R^{\mathbb{R}^n})}{w_{n-1}R^{n-1}} = \lim_{n\to\infty} \frac{\text{Area}(\Sigma_u \cap B_{R/r_n}^{\mathbb{R}^n})}{w_{n-1}(R/r_n)^{n-1}} = \Theta_\infty.$$

Since $r_j\Sigma_u$ is also minimizing, by station varifold theory or Integral currents theory, we could see that Σ_∞ is also minimizing. Because Σ_∞ has constant density and satisfy the monotonicity formula, by the same argument as before, we can see that Σ_∞ is a cone.

But existence of such Σ would imply that there exists nonflat volume minimizing cone C_1^m, $m \leq n-q$ with an isolated singularity at 0 by the splitting theorem of De Giorgi, which contradicts with the following theorem. □

Given $\Sigma^{k-1} \subset S^{n-1}$, the **cone** based on Σ is defined as

$$C(\Sigma) = \{\lambda x : \ x \in \Sigma, \ \lambda \geq 0\}. \tag{14.2}$$

Theorem 14.3 (Simon, 1968)**.** *Every Area minimizing hypersurface in* $\mathbb{R}^n (3 \leq n \leq 7)$ *is flat.*

Proof. Let $\Sigma' = \Sigma \cap S^n(1)$, and assume that $\Sigma = C(\Sigma') \subset \mathbb{R}^{n+1}$ is a regular minimal hypercone(i.e. smooth away from 0). Let $A = (h_{ij})$ be the second fundamental forms. We have (c.f. (10.3))

$$|A|L_\Sigma|A| = |\nabla A|^2 - |\nabla|A||^2.$$

We then claim:

$$|\nabla A|^2 - |\nabla |A||^2 \geq \frac{2|A|^2(x)}{|x|^2}$$

Proof of Claim:

Let $e_1, ...e_{n-1}, e_n$ be orthornormal basis of Σ, s.t. $e_1, ..., e_n \in T\Sigma'$, and $e_n = \frac{x}{|x|}$. Then

$$h_{ij,n} = \nabla_{e_n} h_{ij} = \frac{\partial}{\partial r} h_{ij}(x) = \frac{\partial}{\partial r} h_{ij}(r\frac{x}{|x|}) = \frac{\partial}{\partial r}\frac{h_{ij}(x/|x|)}{r}$$

$$= -\frac{1}{r^2} h_{ij}(\frac{x}{|x|}) + \frac{1}{r}\frac{\partial}{\partial r} h_{ij}(\frac{x}{|x|}) = -\frac{1}{r^2} h_{ij}(\frac{x}{|x|}) = -\frac{1}{r} h_{ij}(x)$$

Moreover, $h_{in} = \langle \nabla_{e_n} e_i, v\rangle = 0$ since $\nabla_{e_n} e_i = 0$.

$$\begin{aligned} |\nabla A|^2 - |\nabla |A||^2 &= \sum_{i,j,k=1}^{n} h_{ij,k}^2 - \frac{\sum_{i,j,k=1}^{n} (h_{ij} h_{ij,k})^2}{\sum_{s,t=1}^{n} |h_{st}|^2} \\ &= \frac{1}{|A|^2}((\sum_{i,j,k=1}^{n} h_{ij,k}^2)(\sum_{s,t=1}^{n} |h_{st}|^2) - \sum_{i,j,k=1}^{n} (h_{ij} h_{ij,k})^2) \\ &= \frac{1}{2|A|^2} \sum_{i,j,r,s,k=1}^{n} (h_{rs} h_{ij,k} - h_{ij} h_{rs,k})^2 \\ &\geq \frac{2}{|A|^2}(\sum_{j,r,s,k=1}^{n} (h_{rs} h_{nj,k} - h_{nj} h_{rs,k})^2)(\text{By symmetry.}) \\ &= \frac{2}{|A|^2}(\sum_{j,r,s,k=1}^{n} (h_{rs} h_{nj,k}) = \frac{2}{|A|^2}(\sum_{j,r,s,k=1}^{n} (h_{rs} h_{jk,n}) \\ &= \frac{2|A|^2}{|x|^2} \end{aligned}$$

By stability, we have

$$\int_\Sigma \phi L \phi \leq 0.$$

Plug $|A|\phi$ in, we have

$$\int_\Sigma \phi^2 |A| L |A| \leq \int_\Sigma |A|^2 |\nabla \phi|^2.$$

By our claim, we can see that

$$2\int_\Sigma \frac{|A|^2}{r^2}\phi^2 \leq \int_\Sigma |A|^2 |\nabla\phi|^2.$$

By multiply a suitable cut-off function, we can prove that the above inequality still holds true for $\phi \in Lip(\mathbb{R}^{n+1})$ provided

$$\int_\Sigma \frac{|A|^2}{r^2}\phi^2 < \infty$$

Our next step is to find a suitable $\phi \in Lip(\mathbb{R}^{n+1})$, s.t.

$$(2-\epsilon)\int_\Sigma \frac{|A|^2}{r^2}\phi^2 \geq \int_\Sigma |A|^2|\nabla\phi|^2,$$

for some $\epsilon > 0$. Hence, $|A| \equiv 0$, which means that Σ is actually a plane.

Since

$$\begin{aligned}\int_\Sigma \frac{|A|^2}{r^2}\phi^2 &= \int_0^\infty \int_{\Sigma\cap S^n(r)} \frac{|A|^2}{r^2}\phi^2 dr \\ &= \int_0^\infty r^{n-1}\int_{\Sigma\cap S^n(1)} \frac{|A|^2}{r^4}\phi^2 dr\end{aligned}$$

If we take

$$\phi(x) = \max\{1, |x|\}^{1-\frac{n}{2}-2\epsilon}|x|^{1+\epsilon},$$

then

$$\int_\Sigma \frac{|A|^2}{r^2}\phi^2 < \infty,$$

provided $n \geq 2$. Also, we can see that

$$|\phi(x)| = \begin{cases} |x|^{2-\frac{n}{2}-\epsilon}, \text{when } |x| \text{ is big enough,} \\ |x|^{1+\epsilon}, \text{when } |x| \text{ is small,}\end{cases}$$

and by directly calculation, we have

$$|\nabla\phi(x)| \leq \begin{cases} |2-\frac{n}{2}-\epsilon|\frac{\phi(x)}{|x|}, \text{when } |x| \text{ is big enough,} \\ (1+\epsilon)\frac{\phi(x)}{|x|}, \text{when } |x| \text{ is small.}\end{cases}$$

Hence,

$$\begin{aligned}\int_\Sigma |A|^2|\nabla\phi|^2 &= \int_0^\infty r^{n-1}\int_{\Sigma\cap S^n(1)} \frac{|A|^2}{r^2}|\nabla\phi|^2 \\ &\leq \max\{1+\epsilon, |2-\frac{n}{2}-\epsilon|\}\int_0^\infty r^{n-1}\int_{\Sigma\cap S^n(1)} \frac{|A|^2}{r^4}|\phi|^2 \\ &\leq (2-\epsilon)\int_\Sigma \frac{|A|^2}{r^2}\phi^2,\end{aligned}$$

provided $n \leq 7$. □

Proposition 14.4. *$C(\Sigma)$ is minimal $\Longleftrightarrow$ Σ is minimal in S^{n-1} $\Longleftrightarrow$ $\Delta_\Sigma x^i + (k-1)x^i = 0$, $i = 1, \cdots, n$, where $\{x^1, \cdots, x^n\}$ is coordinates of $\mathbb{R}^n$.*

Proof. Given $\vec{X} = (x^1, \cdots, x^n)$, then $C(\Sigma)$ is minimal $\Longleftrightarrow$ $\Delta_{C(\Sigma)}\vec{X} = 0$. Take a o.n. basis $\{e_1, \cdots, e_{k-1}\}$ for $T\Sigma$, and $e_k = \vec{X}/|\vec{X}|$, then $\{e_1, \cdots, e_k\}$ is an o.n. basis for $C(\Sigma)$. Then

$$\Delta_{C(\Sigma)}\vec{X} = \sum_{i=1}^{k-1} e_i e_i \vec{X} + e_k e_k \vec{X} - \sum_{i=1}^{k} (\nabla_{e_i} e_i)^{TC(\Sigma)} \vec{X}.$$

Here $\nabla_{e_k} e_k = \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0$, with $r = |\vec{X}|$ the radial function. $\nabla_{e_i} e_i \cdot \vec{X} = -e_i \cdot (\nabla_{e_i} \vec{X}) = -e_i \cdot e_i = -1$, hence $(\nabla_{e_i} e_i)^{TC(\Sigma)} = (\nabla_{e_i} e_i)^{T\Sigma} + (\nabla_{e_i} e_i \cdot \vec{X})\vec{X} = (\nabla_{e_i} e_i)^{T\Sigma} - \vec{X}$. So

$$\Delta_{C(\Sigma)}\vec{X} = \Delta_\Sigma \vec{X} + (k-1)\vec{X}.$$

So we prove the equivalence of the first and the third conclusion.

Now Σ is minimal in S^{n-1} if and only if $\vec{H}_\Sigma = \sum_{i=1}^{k-1} (\nabla_{e_i} e_i)^{TS^{n-1}}$, hence $(\sum_{i=1}^{k-1} (\nabla_{e_i} e_i))$ lies in the normal direction of S^{n-1}. So $\vec{H}_{C(\Sigma)} = (\sum_{i=1}^{k-1} (\nabla_{e_i} e_i) + \nabla_{e_k} e_k)^{\perp(\Sigma)} = \sum_{i=1}^{k-1} (\nabla_{e_i} e_i)^{\perp(\Sigma)} = 0$. □

Now we can give some counter example of Theorem 14.3 when $n \geq 7$:

Clifford Hypersurfaces: Given $S^p(r_1) \subset \mathbb{R}^{p+1}$ and $S^q(r_2) \subset \mathbb{R}^{q+1}$, take

$$\Sigma = S^p(r_1) \times S^q(r_2) \subset \mathbb{R}^{p+q+1}.$$

Then $\Sigma \subset S^{p+q+1} \Longleftrightarrow r_1^2 + r_2^2 = 1$. Given $(\vec{x}, \vec{y}) \in \Sigma$, where $\vec{x} \in S^p$, and $\vec{y} \in S^q$, hence $\Delta_\Sigma \vec{x} = -\frac{p}{r_1^2}\vec{x}$, and $\Delta_\Sigma \vec{y} = -\frac{q}{r_2^2}\vec{y}$. Hence

$$\Sigma^{p+q} \subset S^{p+q+1} \text{ is minimal, } \Longleftrightarrow \frac{p}{r_1^2} = \frac{q}{r_2^2} = p + q.$$

Hence such class of Σ form lots of examples of minimal suffices in S^n and hence minimal cones in $\mathbb{R}^{n+1}$.

Example 14.5. *1.For $p = 3, q = 3$: $C\big(S^3(1/\sqrt{2}) \times S^3(1/\sqrt{2})\big)$ is <u>stable</u>, and area minimizing;*

2. For $p = 1, q = 5$: $C\big(S^1(1/\sqrt{6}) \times S^5(\sqrt{5/6})\big)$ is <u>stable</u>, but not area minimizing.

Chapter 15

The Topology of Minimal surface

In this chapter, we assume that $\Sigma^2 \hookrightarrow S^3(1) \subset \mathbb{R}^4$ is a closed minimal surface.

Lemma 15.1. *We always have $K^\Sigma \leq 1$, and $K^\Sigma = 1$ iff $|A| \equiv 0$.*

Proof. Since $Ric^{S^3}(v,v) + |A|^2 = \frac{1}{2}(R^{S^3} - R^\Sigma + |A|^2)$, i.e.

$$2 + |A|^2 = \frac{1}{2}(6 - 2K + |A|^2).$$

Consequently, $K = \frac{2-|A|^2}{2} \leq 1$, and $K = 1$ iff $|A| \equiv 0$. $\square$

Lemma 15.2. *Let $\Sigma^2 \hookrightarrow (M^3, g)$ is minimal surface, then there exist $f : \Sigma \hookrightarrow M$, s.t. f is conformal and harmonic.*

Theorem 15.3 (Hopf-Almgren)**.** *Σ is immersed minimal surface of $S^3(1)$, then if the genus $g(\Sigma)$ of Σ is zero, then*

$$\Sigma \cong S^2(1) \hookrightarrow S^3(1).$$

Sketch of Proof. Let $f : S^2 \mapsto S^3(1)$ be conformal and harmonic. Consider the Hopf Differential $w = A(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z})dz \otimes dz$. Since $H \equiv 0$, we know that w is holomorphic. However, there is no nonzero holomorphic quadratic form in S^2 (cf. [7], Section 1.6). Hence $w = 0$, i.e. $h_{11} = h_{22}$, which means $A = 0$. Therefore, Σ is the equator. $\square$

Lemma 15.4 (Lawson)**.** *Let $\Sigma \subset S^3(1)$. If $g(\Sigma) = 1$, then Σ has no umbilical point iff $|A| > 0$.*

Proof. Σ is umbilical at $p \in \Sigma$ iff $w(p) = 0$. However, the zeros of w is $4g - 4$. □

Theorem 15.5 (Brendle [8]). *Suppose that $F : \Sigma \to S^3$ is an embedded minimal torus in S^3. Then F is congruent to the Clifford torus.*

Sketch of Proof. We just need to prove $\Psi = \frac{1}{\sqrt{2}}|A| = 1$.

Let

$$\kappa = \sup_{x,y\in\Sigma,\, x\neq y} \sqrt{2}\, \frac{|\langle \nu(x), F(y)\rangle|}{|A(x)|\,(1 - \langle F(x), F(y)\rangle)} < \infty,$$

$$Z_\kappa(x,y) = \frac{\kappa}{\sqrt{2}}\, |A(x)|\,(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle$$

Case 1: if $\kappa \leq 1$, then $Z_1(x,y) \geq 0$. For simplicity, let us identify the surface Σ with its image under the embedding F, so that $F(x) = x$. Let us fix an arbitrary point $\bar{x} \in \Sigma$. We can find an orthonormal basis $\{e_1, e_2\}$ of $T_{\bar{x}}\Sigma$ such that $h(e_1, e_1) = \Psi(\bar{x})$, $h(e_1, e_2) = 0$, and $h(e_2, e_2) = -\Psi(\bar{x})$. Let $\gamma(t)$ be a geodesic on Σ such that $\gamma(0) = \bar{x}$ and $\gamma'(0) = e_1$. We define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = Z(\bar{x}, \gamma(t)) = \Psi(\bar{x})\,(1 - \langle \bar{x}, \gamma(t)\rangle) + \langle \nu(\bar{x}), \gamma(t)\rangle \geq 0.$$

A straightforward calculation gives $-\gamma''(t) = \gamma(t) + h(\gamma'(t), \gamma'(t))\gamma'(t)$, hence

$$f'(t) = -\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \gamma'(t)\rangle,$$

$$\begin{aligned} f''(t) &= \langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \gamma(t)\rangle \\ &\quad + h(\gamma'(t), \gamma'(t))\,\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \nu(\gamma(t))\rangle, \end{aligned}$$

and

$$\begin{aligned} f'''(t) &= \langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \gamma'(t)\rangle \\ &\quad + h(\gamma'(t), \gamma'(t))\,\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), D_{\gamma'(t)}\nu\rangle \\ &\quad + (D^\Sigma_{\gamma'(t)} h)(\gamma'(t), \gamma'(t))\,\langle \Psi(\bar{x})\,\bar{x} - \nu(\bar{x}), \nu(\gamma(t))\rangle. \end{aligned}$$

In particular, we have $f(0) = f'(0) = f''(0) = 0$. Since $f(t)$ is nonnegative, we conclude that $f'''(0) = 0$. From this, we deduce that $(D^\Sigma_{e_1} h)(e_1, e_1) = 0$. An analogous argument with $\{e_1, e_2, \nu\}$ replaced by $\{e_2, e_1, -\nu\}$ yields $(D^\Sigma_{e_2} h)(e_2, e_2) = 0$. Using these identities and the Codazzi equations, we conclude that the second fundamental form is parallel. In particular, the intrinsic Gaussian curvature of Σ is constant. Consequently, the induced metric on Σ is flat. On the other hand, Lawson [3] proved that the Clifford torus is the only flat minimal torus in S^3. Putting these facts together, the assertion follows.

Case 2: $\kappa > 1$. In order to handle this case, we apply the maximum principle to the function

$$Z(x,y) = \frac{\kappa}{\sqrt{2}}\,|A(x)|\,(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle. \tag{15.1}$$

By definition of κ, the function $Z(x,y)$ is nonnegative for all points $x, y \in \Sigma$. Moreover, after replacing ν by $-\nu$ if necessary, we can find two points $\bar{x}, \bar{y} \in \Sigma$ such that $\bar{x} \neq \bar{y}$ and $Z(\bar{x}, \bar{y}) = 0$. Moreover, we claim that:

$$\Omega = \{\bar{x} \in \Sigma : \text{there exists a point } \bar{y} \in \Sigma \setminus \{\bar{x}\} \text{ such that } Z(\bar{x}, \bar{y}) = 0\}$$

is non-empty and open. Moreover, $\nabla\Psi(\bar{y}) \equiv 0, \forall \bar{y} \in \Omega$. This is because, by some computations and the fact that Z attain its local minimal at $(\bar{x}, \bar{y})$, we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) \\ &= -\frac{\kappa^2 - 1}{\kappa}\,\frac{\Psi(\bar{x})}{1 - \langle F(\bar{x}), F(\bar{y})\rangle} \sum_{i=1}^{2} \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle^2 \leq 0, \end{aligned} \tag{15.2}$$

This gives

$$\Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle = 0.$$

Also, since Z attain its local minimal at $(\bar{x}, \bar{y})$, by some computation, we have At the point $(\bar{x}, \bar{y})$, we have

$$\begin{aligned} 0 = \frac{\partial Z}{\partial x_i}(\bar{x},\bar{y}) &= \frac{\partial \Phi}{\partial x_i}(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle) \\ &\quad - \Phi(\bar{x})\,\Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle + h_i^k(\bar{x})\,\Big\langle \frac{\partial F}{\partial x_k}(\bar{x}), F(\bar{y}) \Big\rangle \qquad (15.3) \\ &= \kappa\,\frac{\partial \Psi}{\partial x_i}(\bar{x})\,(1 - \langle F(\bar{x}), F(\bar{y})\rangle), \qquad (15.4) \end{aligned}$$

which imply $\nabla\Psi = 0$ in Ω.

Since Ω is open, it follows from the claim that $\Delta_\Sigma \Psi(\bar{x}) = 0$ for each point $\bar{x} \in \Omega$. Since

$$\Delta_\Sigma \Psi - \frac{|\nabla\Psi|^2}{\Psi} + (|A|^2 - 2)\,\Psi = 0,$$

we can see that $\phi \equiv 1$ in Ω. Using standard unique continuation theorems for solutions of elliptic partial differential equations (see e.g. [9]), we conclude that $\Psi(x) = 1$ for all $x \in \Sigma$. Consequently, the Gaussian curvature of Σ vanishes identically. As above, it follows from a result of Lawson [3] that F is congruent to the Clifford torus. $\square$

Chapter 16

Closed Minimal Surfaces in S^3

Let $\Sigma^2 \hookrightarrow S^3$ be a immersed minimal surface, then the Jacobi operator

$$L_\Sigma \phi = \Delta\phi + (|A|^2 + Ric^{S^3}(\nu,\nu))\phi = \Delta\phi + (2+|A|^2)\phi,$$

and

$$Q(\phi,\phi) := -\int_\Sigma \phi L_\Sigma \phi = \int_\Sigma |\nabla\phi|^2 - (2+|A|^2)\phi^2,$$

where ν is unit normal vector fields on Σ.

Recall that Morse index $Ind(L_\Sigma) :=$ number of negative eigenvalue of L_Σ with multiplicity. Moreover, since $Q(1,1) < 0$, we can see that $Ind(L_\Sigma) \geq 1$. Let $\lambda_1(\Sigma)$ be the first eigenvalue of L_Σ.

Proposition 16.1. *Let $S^2 \hookrightarrow S^3$ be a equator, i.e. $|A| = 0$, then it's obvious S^2 is minimal, also, we have $\lambda_1(\Sigma) = -2$ and $\lambda_2(\Sigma) = 0$.*

Proof. Since function 1 is a eigenfunction with eigenvalue -2, and 1 doesn't change sign, by Proposition 6.11, $\lambda_1(S^2) = -2$. By Theorem 5.1 in [10], we can see that $\lambda_2(S^2) \geq 0$. By Proposition 14.4, we have $\lambda_2(S^2) = 0$. □

Proposition 16.2. *Let $T_c \hookrightarrow S^3$ be the Clifford torus. Then $Ind(T_c^2) = 5$ with $\lambda_1 = -4$ with multiplicity 1 and its eigenfunction is $u_1 = const$; $\lambda_2 = -2$, with multiplicity 4 and its eigenfunctions are: $\cos(\sqrt{2}\theta_1)$, $\sin(\sqrt{2}\theta_1)$, $\cos(\sqrt{2}\theta_2)$ and $\sin(\sqrt{2}\theta_2)$; $\lambda_3 = 0$.*

Proof. $L_{T_c}\phi = \Delta\phi + 4\phi$. This is because, $K = 1 - \frac{|A|^2}{2} \equiv 0$. Hence, 1 is a eigenfunction with eigenvalue -4, and since 1 doesn't change sign, $\lambda_1 =$

-4. By Theorem 5.1 in [10] again, $\lambda_2 \geq -2$. By Proposition 14.4 again, it's easy to check that $\cos(\sqrt{2}\theta_1)$, $\sin(\sqrt{2}\theta_1)$, $\cos(\sqrt{2}\theta_2)$ and $\sin(\sqrt{2}\theta_2)$ are eigenfunctions with eigenvalue -2, hence $\lambda_2 = -2$, or it's straightforward to verify that the first three eigenvalues of Δ_{T_c} on $T_c \cong S^1 \times S^1$ are $0, 2$ (with multiplicity 4), 4. Thus, $\lambda_2 = -2, \lambda_3 = 0$. □

Proposition 16.3. *Let $\Sigma^2 \hookrightarrow S^3$ be immersed minimal surfaces, let ν be the unit normal vector field on Σ, then we have*

$$\Delta_\Sigma \nu + |A|^2 \nu = 0.$$

Therefore, Σ always have eigenvalue -2.

Proof. Consider $C(\Sigma) \hookrightarrow \mathbb{R}^4$, then for any $w \in R^4$, we have $C_t(\Sigma) = C(\Sigma) + tw$ is isometry. Hence, $\langle w, \nu \rangle$ is a Jacobi field, which means

$$\Delta_\Sigma \langle w, \nu \rangle + |A|^2 \langle w, \nu \rangle = 0.$$

Equivalently,

$$\Delta_\Sigma \nu + |A|^2 \nu = 0.$$

□

Theorem 16.4. *Let $\Sigma^2 \hookrightarrow S^3$ be immersed minimal surfaces, if Σ is not a equator, then $Ind(\Sigma) \geq 5$. When $Ind(\Sigma) = 5$, $\Sigma \cong T_c^2$.*

Proof. Let $\nu = (\nu_1, ..., \nu_4)$ denotes the unit normal vector to Σ. If $\lambda_1 = -2$, then by Proposition 6.11, we have $a_1, a_2, a_3 \in \mathbb{R}$, s.t. $\nu_1 = a_1\nu_2 = a_2\nu_3 = a_3\nu_4$. Since $\Sigma \ni x \cdot \nu = 0$, we have $x_1 + a_1x_2 + a_2x_3 + a_3x_4 = 0$, which mean Σ lie in a plane pass through 0. Thus, Σ is a equator, which is impossible.

Moreover, Since Σ is not total geodesic (otherwise Σ is an equator), we can see v_1, v_2, v_3, v_4 are linear independent. Hence, by now we have prove $Ind(\Sigma) \geq 5$.

Suppose now that the Jacobi operator L has exactly five negative eigenvalues. Let ρ denote the eigenfunction associated with the eigenvalue λ_1. Note that ρ is a positive function. We consider a conformal transformation $\psi : S^3 \to S^3$ of the form

$$\psi(x) = a + \frac{1 - |a|^2}{1 + 2\langle a, x \rangle + |a|^2}(x + a),$$

where a is a vector $a \in \mathbb{R}^4$ satisfying $|a| < 1$. We can choose the vector a in a such a way that

$$\int_\Sigma \rho\, \psi_i(x) = 0$$

for $i \in \{1, 2, 3, 4\}$, where $\psi_i(x)$ denotes the i-th component of the vector $\psi(x) \in S^3 \subset \mathbb{R}^4$.

By assumption, L has exactly five negative eigenvalues. In particular, L has no eigenvalues between λ_1 and -2. Since the function ψ_i is orthogonal to the eigenfunction ρ, we conclude that

$$\int_\Sigma (|\nabla^\Sigma \psi_i|^2 - |A|^2\, \psi_i^2) = \int_\Sigma \psi_i\, (L\psi_i + 2\psi_i) \geq 0 \tag{16.1}$$

for each $i \in \{1, 2, 3, 4\}$. On the other hand, the conformal invariance of the Willmore functional implies that

$$\sum_{i=1}^4 \int_\Sigma |\nabla^\Sigma \psi_i|^2 = 2\,\text{area}(\psi(\Sigma)) \leq 2\,\mathscr{W}(\psi(\Sigma)) = 2\,\mathscr{W}(\Sigma) = 2\,\text{area}(\Sigma). \tag{16.2}$$

where

$$\mathscr{W}(\Sigma) = \int_\Sigma 1 + \frac{H^2}{4} = \int_\Sigma K + \frac{|A|^2}{2}$$

is conformal invariant. Moreover, it follows from the Gauss-Bonnet theorem that

$$\sum_{i=1}^4 \int_\Sigma |A|^2\, \psi_i^2 = \int_\Sigma |A|^2 = 2\int_\Sigma (1 - K) \geq 2\,\text{area}(\Sigma). \tag{16.3}$$

Combining the inequalities (16.2) and (16.3) gives

$$\sum_{i=1}^4 \int_\Sigma (|\nabla^\Sigma \psi_i|^2 - |A|^2\, \psi_i^2) \leq 0. \tag{16.4}$$

Putting these facts together, we conclude that all the inequalities must, in fact, be equalities. In particular, we must have $\mathscr{W}(\psi(\Sigma)) = \text{area}(\psi(\Sigma))$. Consequently, the surface $\psi(\Sigma)$ must have zero mean curvature. This implies that $\langle a, \nu \rangle = 0$ at each point on Σ. Since Σ is not totally geodesic, it follows that $a = 0$. Furthermore, since $\int_\Sigma \rho\, \psi_i = 0$ and $\int_\Sigma (|\nabla^\Sigma \psi_i|^2 - |A|^2\, \psi_i^2) = 0$, we conclude that the function ψ_i is an eigenfunction of the Jacobi operator with eigenvalue -2. Consequently, $\Delta_\Sigma \psi_i + |A|^2\, \psi_i = 0$ for each $i \in \{1, 2, 3, 4\}$. Since $a = 0$, we conclude that $\Delta_\Sigma x_i + |A|^2\, x_i = 0$. Since $\Delta_\Sigma x_i + 2\, x_i = 0$, we conclude that $|A|^2 = 2$ and the Gaussian curvature of Σ vanishes. This implies that Σ is the Clifford torus. □

Chapter 17

General Relativity

(S^{n+1}, g) is used to model the space-time, where S^{n+1} is an $n+1$ dimensional smooth oriented manifold, g is a Lorentz metric with signature $(-1, 1, ..., 1)$.

Example 17.1 (Flat model)**.** $\mathbb{R}^{n,1}$

- $x_0, x_1, ..., x_n$ *are coordinates*
- $g = -dx_0^2 + \sum_{i=1}^n dx_i^2$ *is the Lorentzian metric;*
- $\langle v, w \rangle = -v_0 w_0 + \sum_{i=1}^n v_i w_i$*;*

We have 3 *types of vectors:*

- *space-like:* $\langle v, v \rangle > 0$*;*
- *time-like:* $\langle v, v \rangle < 0$*;*
- *null:* $\langle v, v \rangle = 0$.

Let $H^n \subset R^{n,1}$ *be a plane, then there exists a unique* $v \neq 0$ *up to scale, such that* $H = \{w : \langle v, w \rangle = 0\}$.

- H *is space-like if* v *is time-like, then* $g|_H$ *has positive signature;*
- H *is time-like if* v *is space-like, then* $g|_H$ *has Lorentz signature;*
- H *is null if* $v \in H$ *and* $\langle v, v \rangle = 0$, *then* $g|_H$ *is degenerate.*

$M^n \subset \mathbb{R}^{n,1}$ *is space-like if* $T_x M$ *is space like for all* $x \in M$.

17.1 Einstein Equations

The Einstein equation is given by

$$Ric^n - \frac{1}{2}R^n g = T, \tag{17.1}$$

where Ric and R are Ricci curvature and Scale curvature respectively. When $T = 0$, we call (17.1) Vacuum Einstein Equation(VEE).

Example 17.2 (Schwartzchild)**.** *For any* $m > 0$, $(\mathbb{R}^3/\{0\} \times \mathbb{R}, g)$, *and*

$$g = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\sigma,$$

where $r = |x|, x = r\sigma$.

Given v a time-like unit vector s.t. $\langle v, v\rangle - 1$, then

- $T(v, v)$ is the observed energy density of observer;
- $T(v, \cdot)^*$ is energy-momentum density vector.

Dominant Energy Condition(DEC): $\forall v$ unit time-like vector $\Rightarrow T(v, .)^*$ is forward pointing time-like or null. Given $v = e_0, e_1, ..., e_n$ an o.n. basis, and let $T_{ab} = T(e_a, e_b)$, then (DEC) requires $T_{00} \geq \sqrt{\sum_{i=0}^n} T_{0i}$.

17.2 Initial value problem

The initial data is modeled by a triple (M^n, g, h), where g is a Riemmanian metric, and h a symmetric $(0, 2)$ tensor.

Our problem is: Given initial data, find a local evolution (S^{n+1}, eta) of VEE with $M \subset S, g = \eta|_M; h =$ second fundamental form.

We need the constraint equations (CE) for g, h so that such (S, η) exists:

$$\begin{cases} T_{00} = \mu = \frac{1}{2}(R + (tr_g)^2 - |h|^2), \\ T_{0i} = J = div_g(h - (tr_g h)g). \end{cases}$$

We call T_{00} the observed energy density, $T_(0j)$ observed momentum density.

The vacuum constraint equations (VCE) are given by:

$$\begin{cases} 0 = \frac{1}{2}(R + (tr_g h)^2 - |h|^2), \\ 0 = div_g(h - (tr_g h)g). \end{cases}$$

17.2.1 Derivation of (CE)

Given (S^{n+1}, η) the space-time, let (M^n, g, h) be the initial data set, such that (g, h) are the restriction and second fundamental form of $M \subset (S^{n+1}, \eta)$. Take $\{e_0, e_1, ..., e_n\}$ an o.n. frame of S at $p \in M$, with $e_0 \perp M$ and $e_i \in T_pM$. Consider the Einstein equation

$$R_{ab} - \frac{1}{2}R\eta = T_{ab}, 0 \leq a, b \leq n.$$

First, by Gauss Equation, $M \subset S$, $X, Y, Z, W \in TM$,

$$R^M(X,Y,Z,W) = R^S(X,Y,Z,W) + \langle II(X,Z), II(YW)\rangle - \langle II(X,W), II(Y,Z)\rangle,$$

where $II(X,Y) = (D_XY)^\perp = h(X,Y)e_0$. Plug in e_i, e_j, e_k, e_l,

$$R^M_{ijkl} = R^S_{ijkl} - h_{ik}h_{jl} + h_{il}h_{jk}.$$

Summing over i, k and j, l respectively,

$$R^M = \sum_{i,j=1}^{n} R^M_{ijij} = \sum_{i,j=1}^{n} R^S_{ijij}(tr_g h)^2 + |h|^2 :$$

Now

$$\sum_{i,j=1}^{n} R^S_{ijij} = \sum_{i,j=1}^{n} R^S_{ijij} - 2\sum_{i=1}^{n} R^S_{0i0i} + 2\sum_{i=1}^{n} R^S_{0i0i} = 2(R^S_{00} - \frac{1}{2}R^S g_{00}) = 2T_{00}.$$

Similarly By Codazzi Equation, we get the equation for T_{0i}.

Theorem 17.3. *There is a one-one corresponding between (M, g, h) satisfy (VCE) and its maximal hyperbolic evolution (up to diffeomorphism).*

17.3 Asymptotical Flatness

(M^n, g, h) is is an Initial data set satisfying the (CE). Roughly speaking, we say (M, g, h) is asymptotically flat (AF) if outside a compact set $K \subset M$, $M\backslash K$ is diffeomorphic to $\mathbb{R}^n \backslash B^n$. Moreover denote $\{x_1, ..., x_n\}$ to be the coordinates on $R^n\backslash B$, then we assume $g \in C^{2,\alpha}(\mathbb{R}^n\backslash B)$ and

$$g_{ij}(x) = \delta_{ij} + \Gamma,$$

where $|\Gamma| \in O(r^{-1/2}), |\partial\Gamma| = O(r^{-3/2}), |\partial^2\Gamma| = O(r^{-5/2})$.

Definition 17.4 (Weighted Sobolev space)**.** *We say* $f \in W^{k,p}_{-q}(M^3), q > 1/2, p > 3$, *if*

$$\|f\|_{W^{k,p}_{-q}} = \|f\|_{W^{2,p}(K)} + (\int_{\mathbb{R}^3 \setminus B} \sum_{|\beta| \le k} (|x|^{q+|\beta|} |\partial^\beta f|)^p \frac{dx}{|x|^n})^{1/p} < \infty.$$

Definition 17.5. *(M^n, g, h) is called asymptotically flat (with one end), if:*
1. $K \subset M$ is compact, such that $M \setminus K$ is diffeomorphic to $R^n \setminus B$. Let $x_1, ..., x_n$ be the local coordinates given by $R^n \setminus B$.

2. $g \in C^{2,\alpha}(M), h \in C^{1,\alpha}(M)$, and

$$g_{ij} = \delta_{ij} + \gamma_{ij}, \gamma \in W^{2,p}_{-q}(M), q > \frac{n-2}{2}, p > n$$

and

$$h \in W^{1,p}_{-1-q}(M)$$

3. The mass density μ and momentum density J in (CE) satisfy: $\mu, J \in C^{0,\alpha}_{-q_0}(M)$, $q_0 > n$.

Example 17.6 (Schwartzchild Solution (SC))**.** *Consider $(\mathbb{R}^0 \setminus \{0\}, g, 0)$, where*

$$g_{ij} = (1 + \frac{m}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta.$$

When $m > 0$, $u = 1 + \frac{m}{2|x|^{n-2}}$ is harmonic on $\mathbb{R} \setminus \{0\}$. Moreover $R_g = -\frac{4n-4}{n-2} u^{-\frac{n+2}{n-2}} (\Delta u) = 0$, so it is vacuum.

Definition 17.7. *Given (M, g, h) satisfy AF, the ADM mass is*

$$m_{ADM} = \frac{1}{16} \lim_{r \to \infty} \int_{\partial S^r} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \nu^j d\sigma$$

Example 17.8. *For (SC), $m_{ADN} = m$.*

Theorem 17.9 (Positive mass theorem, Schoen-Yau 1978, Witten 1980)**.** *Let (M^n, g) be AF with $R_g \ge 0$. Then we have $m_{ADM} \ge 0$. Moreover, $m = 0$ iff $(M^n, g) \cong (\mathbb{R}^n, \delta_{ij})$*

Proof. See section 5.5 in `http://www.mit.edu/~cmad/Papers/MinimalSubmanifoldNotes.pdf`. □

Chapter 18

Geometric Measure Theory and Minimal Surfaces

Let $U \subset M^n$ be open, let $V^k(U)$ denote the C^∞ section of $k-$form with compact support in U. We can endow the locally smooth topology on $V^k(U)$. For any $\omega \in V^k(U)$, denote $\| \omega \|_{L^\infty} = \sup_{x\in U} \langle \omega(x), \omega(x)\rangle^{1/2}$.

Definition 18.1. *A $k-$dimensional current $V_k(U)$ in U is just a bounded linear functional on $V^k(U)$.*

Remark 18.2. *1. When $k = 0$, $V_0(U) =$ is just the Schwartz distribution.*

2. If N is a $k-$dimensional submanifold in U, then we regard it as a element in $V_k(U)$ as following

$$\langle [N], \omega\rangle = \int_N \omega.$$

Moreover, if we have a family $k-$dimensional submanifolds of N_t in U, by choose a good function $f(t)$, we can define a current

$$\int (\int_{N_t} \omega) f(t) dt.$$

Given a current $T \in V_k(U)$, we could define its boundary ∂T by

$$\partial T(w) = T(dw), \forall w \in V^{k-1}(U).$$

Moreover, it's easy to see that $\partial^2 T = 0$.

Definition 18.3. *The mass of $T \in V_k(U)$ is defined by*

$$\mathcal{M}_U(T) = \sup_{\omega \in V^k(U), \|\omega\|_{L^\infty} \le 1} T(\omega).$$

For any $T \in V_k(U)$, by Riesz representation theory (Compare Theorem 4.1 in [11]), we could always find $\tau \in V^k(U)$, a Borel measure u_T on U, s.t. for any $\omega \in V^k$, we have

$$T(\omega) = \int_U \langle \tau, w \rangle du_T.$$

Moreover, for any open set $W \subset U$, we have

$$u_T(W) = \sup_{\omega \in V^k(W), \|\omega\|_{L^\infty} \leq 1} T(\omega) = \mathcal{M}_W(T).$$

Now we are able to define the restriction. Let A be a u_T-measurable subset of U, then we define the restriction of T on A, denoted by $T|_A$ by

$$T|_A(\omega) = \int_A \langle w, \tau \rangle du_T.$$

In addition, we define the support $spt(T)$ of $T \in V_k(U)$ by

$$spt(T) = spt(u_T) \subset U.$$

Definition 18.4. *We say $T_j \in V_k(U)$ converge to T weakly, which is denoted by $T_j \rightharpoonup T$, if for any $\omega \in V^k(U)$,*

$$\lim_{j \to \infty} T_j(\omega) = T(\omega).$$

Lemma 18.5 (Compactness)**.** *If $\sup_k \mathcal{M}_W(T_i) < \infty, \forall W \subset\subset U$. Then there exist a subsequence $T_{i'}$ of T_i, s.t.*

$$T_{i'} \rightharpoonup T,$$

for some $T \in V_k(U)$.

Now we are considering the locally Lipchitz submanifolds.

Definition 18.6. *A sub $M \subset U$ is called countably $k-$retifiable if*

$$M \subset \cup_{j=0}^{\infty} M_0,$$

s.t.

$$\mathcal{H}^k(M_0) = 0, M_j \subset F_j(A_j),$$

where A_j is a domain in $\mathbb{R}^k$ and $\mathcal{H}^k$ is $k-$Hausdorff measure (See [11] for a detail).

Proposition 18.7 (Criterion for rectifiable)**.** *M is rectifiable iff for $\mathcal{H}^k-$a.e. $x \in M$, there exist unique k dimensional tangent plane P, s.t.*

$$\int_{\eta_{x,\lambda}(M)} f(y)\theta(x+\lambda y)d\mathcal{H}^k(y) \to \theta(x)\int_P f(y)d\mathcal{H}^k(y), \lambda \to 0^+,$$

where θ is a locally $\mathcal{H}^k-$integrable function on M, and $\eta_{x,\lambda}(y) = \frac{y-x}{\lambda}$

Definition 18.8. *We say T is an inter-rectifiable $k-$current, if*

$$T(\omega) = \int_M \langle \omega, \xi \rangle \theta(x) d\mathcal{H}^k,$$

where M is a $\mathcal{H}^k-$measurable countably $k-$rectifiable subset of U, $\xi : M \mapsto \wedge_k(\mathbb{R}^n)$ s.t. for $\mathcal{H}^k-$a.e. on M, $\xi(x) = \tau_1 \wedge ... \wedge \tau_k$, $\{\tau_1, ...\tau_k\}$ are orientable basis of $P = T_xM$, $\theta : M \mapsto \mathbb{Z}$ is locally $\mathcal{H}^k-$integrable.

Such a T is called integrable if

$$\mathcal{M}_W(\partial T) < \infty, \forall W \subset\subset U. \tag{18.1}$$

Example 18.9. *We would like to explain that condition 18.1 is nontrivial by this example. More specific, $\mathcal{M}(T) < \infty$ doesn't imply $\mathcal{M}(\partial T) < \infty$.*

Let $T = \{x \in \mathbb{R}^2 : |x| = 2^{-n}, n \in \mathbb{Z}^+\}$, $T' = \{x \in \mathbb{R}^2 : |x| = 2^{-n}, n \in \mathbb{Z}^+\} \cap (a,b) \in \mathbb{R}^2 : B > 0$.

Then, it's easy to see that

$$\mathcal{M}(T) = 2, \partial T = 0,$$

but

$$\mathcal{M}(T') = 1, \mathcal{M}(\partial T') = \infty.$$

Theorem 18.10. *If $T_j \subset V_k(U)$ integrable, and*

$$\sup\{\mathcal{M}_W(T_j) + \mathcal{M}_W(\partial T_j)\} < \infty, \forall W \subset\subset U,$$

then there exist T integrable, s.t. $T_j' \rightharpoonup T$ for some subsequence.

Example 18.11. *Let $T_m = \cup_{j=0}^{m-1}[\frac{2j}{2m}, \frac{2j+1}{2m}]$, we have $T_m \rightharpoonup \frac{1}{2}[0,1]$*

Definition 18.12 (Flat Norm)**.** *Let $I_k(U)$ be the space of $k-$dimensional integral current. Given $T_1, T_2 \in I_k(U), W \subset\subset U$, the flat norm*

$$\mathcal{F}(T_1, T_2) := \inf_{S \in I_{k+1}(U)} \{\mathcal{M}(S) + \mathcal{M}(\partial S - T_1 - T_2)\}.$$

Theorem 18.13. *$T_j, T \subset I_k(U)$, $T_j \rightharpoonup T$ iff $\mathcal{F}(T_j, T) \to 0$.*

We now consider the Plateau problem: Given $T^{k-1} = \partial M_0^k \hookrightarrow \mathbb{R}^n$, if there exist an area minimizing M, s.t. $\partial M = T$.

Theorem 18.14. *Let $S \in I_{k-1}(\mathbb{R}^n)$ with compact support, $\partial S = 0$. Then there exist $T \in I_k(\mathbb{R}^n)$, s.t.*

$$\begin{cases} T \text{ has compact support}, \\ \partial T = S, \\ \mathcal{M}(T) \le \mathcal{M}(R), \forall R \in I_k(\mathbb{R}^n), \text{ s.t. } \partial R = S. \end{cases}$$

We could build a homology theory for I_k.

Let $(M^n, g) \hookrightarrow \mathbb{R}^l$ be a Riemannian manifold, let

$$I_k(M) = \{T \in I_k(\mathbb{R}^l), \operatorname{spt}(T) \subset M\},$$

$$Z_k(M) = \{T \in I_k(M), \partial T = 0\},$$

$$B_k(M) = \{T \in I_k(M), T = \partial S \text{ for some } S \in I_{k+1}(M)\},$$

Then we define

$$H_k(M, \mathbb{Z}) = Z_k(M) \otimes \mathbb{Z} / B_k(M) \otimes \mathbb{Z}$$

Theorem 18.15. *$H(M, \mathbb{Z}) \cong H_{Sing}(M, \mathbb{Z})$, where H_{Sing} is the singular homology.*

Remark 18.16 (Not Sure if it is correct?)**.** *If $[T] \in H_k(M) \neq 0$, then there exist $T_0 \in Z_k(M)$, s.t. $\mathcal{M}(T) \le \mathcal{M}(R), \forall R \in [T]$.*

Definition 18.17 (Varifold)**.** *Let $U \subset \mathbb{R}^l$ be an open set. An varifold V of dimension k in U is a pair (M, θ), where $M \subset U$ is a rectifiable set of dimension k and $\theta : M \mapsto \mathbb{Z}^+$ a Borel map. We can identify a Varifold $V = (M, \theta)$ with an inter-rectifiable $k-$current, as follow*

$$V(\omega) = \int_M \langle \omega, \xi \rangle \theta(x) d\mathcal{H}^k,$$

Hence, we can define $\mathcal{M}(V)$ on V.

Proposition 18.18 (First Variation)**.** *Let $V = (M, \theta)$, and M be a submanifold of $\mathbb{R}^l$, let ϕ^t be the flow of vector field X, then define $\phi^t(V) = (\phi^t(M), \theta \circ \phi^{-t})$, we have*

$$\delta V(X := \frac{d}{dt} \mathcal{M}(\phi^t(M, \theta))|_{t=0} = \int_M \operatorname{div}_M(X) d\mathcal{H}^k,$$

We say V is stationary if $\delta V(X) \equiv 0, \forall X$ with compact support.

Theorem 18.19. *If V is stationary and $\theta = 1, \mathcal{H}^k-$a.e., then there exist $\epsilon = \epsilon(n,k)$, s.t. if $\frac{\mathcal{M}(V|_{B(x,r)})}{w_k r^k} < 1+\epsilon$, then $M|_{B(x,r)}$ is C^∞ embedded minimal submanifold.*

Theorem 18.20. *$T \in I_{k-1}(U)$ is area minimizing and $n \leq 7$, where $U \subset \mathbb{R}^k$, then T is regular.*

Proof. See Section 4 in [12] for a sketch of proof. □

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