MULTIPLICITY ONE CONJECTURE IN MIN-MAX THEORY

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The min-max theory is a powerful tool to find minimal surfaces, which are the mathematical models for soap films. Motivated by Yau's conjecture on minimal surfaces [15], Marques and Neves proposed a program to establish the Morse theory for the Area functional [6, 7, 8], in which they explored the notion of "volume spectrum" introduced by Gromov in 1980s [4]. One of their goals is to understand the key feature of the min-max theory, that is, the Morse index. The long-standing challenge of min-max theory, especially for Marques-Neves's program, was the "*Multiplicity One Conjecture*" [8, 1.2]. The conjecture said that minimal hypersurfaces produced by the min-max theory are always two-sided and have multiplicity one a generic scenario. This conjecture is a natural nonlinear analog of a famous result by Uhlenbeck [12] for the linear "Laplacian spectrum" in 1960s. This conjecture was proved by the author in [16].

Now we start to state the precise result. Let (M^{n+1}, g) be a closed orientable Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. In [1], Almgren proved that the space of mod-2 cycles $\mathcal{Z}_n(M, \mathbb{Z}_2)$ is weakly homotopic the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$; (see also [8] for a simpler proof). Later, Gromov [4], Guth [5], Marque-Neves [7] introduced the notion of volume spectrum as a nonlinear version of spectrum for the area functional in $\mathcal{Z}_n(M, \mathbb{Z}_2)$. In particular, the volume spectrum is a non-decreasing sequence of positive numbers

$$0 < \omega_1(M, g) \leq \cdots \leq \omega_k(M, g) \leq \cdots \to +\infty,$$

which is uniquely determined by the metric g in a given closed manifold M.

By adapting the celebrated min-max theory developed by Almgren [2], Pitts [9] (for $3 \leq (n+1) \leq 6$), and Schoen-Simon [11] (for $n+1 = 7$), Marques-Neves [7, 6] proved that each $\omega_k(M, g)$ is associated with an integral varifold V_k whose support is a disjoint collection of smooth, connected, closed, embedded, minimal hypersurfaces $\{\Sigma_1^k, \cdots, \Sigma_{l_k}^k\}$, such that

$$\omega_k(M, g) = \sum_{i=1}^{l_k} m_i^k \cdot \mathrm{Area}(\Sigma_i^k), \tag{0.1}$$

where $\{m_1^k, \cdots, m_{l_k}^k\} \subset \mathbb{N}$ is a set of positive integers, usually called *multiplicities*.

Our main theorem states that if a component Σ_i^k is not *weakly stable*, then Σ_i^k has to be two-sided and the associated integer multiplicity is identically equal to one, i.e. $m_i^k = 1$. Note that a closed minimal hypersurface Σ is said to be *weakly stable* if it has a 0 as the lowest eigenvalue for the second variation of area; (when Σ is one-sided, one has to pass to its two-sided double cover).

Theorem 0.1. *Given a closed manifold (M^{n+1}, g) of dimension $3 \leq (n+1) \leq 7$, denote $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ as the min-max minimal hypersurfaces associated with volume spectrum. Then every connected component of $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ which is not weakly stable is two-sided and has multiplicity one. That is, if Σ_i^k is not weakly stable, $k \in \mathbb{N}$, $1 \leq i \leq l_k$, then Σ_i^k is two-sided* *and* $m_i^k = 1$, *and*

$$\sum_{i=1}^{l_k} \mathrm{index}(\Sigma_i^k) \leq k.$$

Remark 0.2. Theorem 0.1 is an equivalent formulation of the *Multiplicity One Conjecture* of Marques-Neves [8, 1.2] proved by the author in [16, Theorem A]. Indeed, [16, Theorem A] asserts that for a bumpy metric g, all connected components of $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ are two-sided and have multiplicity one. Theorem 0.1 directly implies [16, Theorem A], as weakly stable minimal hypersurfaces are degenerate and hence do not exist in a bumpy metric. Now we argue that that [16, Theorem A] implies Theorem 0.1. A metric g is called *bumpy* if every closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in Baire sense [13, 14]. For an arbitrary metric g, we can take a sequence of bumpy metrics $\{g_j\}_{j\in\mathbb{N}}$ such that $g_j \to g$ smoothly. We also know that the k-widths $\{\omega_k(M, g_j)\}_{j\in\mathbb{N}}$ converges to $\omega_k(M, g)$ as $j \to \infty$ for each $k \in \mathbb{N}$. Now fix $k \in \mathbb{N}$; for each g_j, the associated min-max minimal hypersurfaces $V_{k,j}$ are all two-sided and have multiplicity one by [16, Theorem A]. By the compactness theorem [10, Theorem A.6], V_k converges up to a subsequence to a limit integral varifold V, such that the support $\mathrm{spt}(V)$ of V is smooth embedded minimal hypersurfaces. Now using [10, Theorem A.6] again, if a connected component of $\mathrm{spt}(V)$ either has multiplicity greater than one or is one-sided, it (or its two-sided double cover when one-sided) has to carry a positive Jacobi field for the second variation of area, and hence it is weakly stable.

Remark 0.3. Recently, Chodosh-Mantoulidis [3] proved this conjecture in dimension three $(n+1) = 3$ for the Allen-Cahn setting; they also proved that the total index is exactly k for their k-min-max solutions when $(n + 1) = 3$. After our results were posted, Marques-Neves finished their program and also proved the same optimal index estimates for $3 \leq (n + 1) \leq 7$ [8, Addendum].

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