Γ a lattice in a finite dimensional \( \mathbb{R} \) vector space \( \mathcal{W} \) (lattice basis \( \sim \to \mathbb{Z}^d \subset \mathbb{R}^d \))

**Definition**

\( q: \Gamma \to \mathbb{C} \) is *quasi-polynomial* if there is a sublattice \( \Gamma' \) with \( \Gamma/\Gamma' \) finite such that restriction of \( q \) to each coset of \( \Gamma' \) is polynomial. (More generally allow \( q \) to be only partially defined.)

\[
q(\gamma) = \sum_g g^{-\gamma} q_g(\gamma)
\]

\( q_g \) polynomial, sum over \( g \in \text{Hom}(\Gamma, U(1)) \) of finite order
Ehrhart’s theorem

- $V$ finite dimensional $\mathbb{R}$ vector space, dimension $d$
- $\Lambda$ rank $d$ lattice in $V$
- $P$ $d$-dimensional closed polytope with vertices in $\Lambda \otimes \mathbb{Q}$

$$N_P(k) = \#(kP \cap \Lambda) = \#(P \cap \frac{1}{k}\Lambda)$$

**Theorem (Ehrhart)**

$N_P(k)$ is a quasi-polynomial function of $k$.

Leading term $\text{vol}_\Lambda(P)k^d$
Around \([Q, R] = 0\)

- \(G\) compact connected Lie group, max torus \(T\), weight lattice \(\Lambda \subset \mathfrak{t}^*\), \(\Lambda_+\) dominant weights
- \((M, \omega, \mu)\) compact Hamiltonian \(G\)-space, compatible a-\(\mathbb{C}\) structure \(J\), prequantum line bundle \(L\)
- \(D = \overline{\partial} + \overline{\partial}^*\) Dolbeault-Dirac operator, \(D_{L^k}\) twist by \(L \otimes k\)

\[
\text{index}_G(D_{L^k}) = \sum_{\lambda \in \Lambda_+} m(k, \lambda) \chi_{\lambda}
\]

\[m: \mathbb{Z}_{>0} \times \Lambda_+ \to \mathbb{Z}\]

**Theorem (Meinrenken-Sjamaar, '99)**

\(\exists\) a finite set of closed polytopes \(P \subset \mathfrak{t}^*\) such that \(m|_{C_P}\) is quasi-polynomial, where

\[
C_P = \{(t, tv) | t > 0, v \in P\} \subset \mathbb{R} \times \mathfrak{t}^*
\]
The cone $C_P$ for $P = [0, 1] \subset t^* = \mathbb{R}$, $\Lambda = \mathbb{Z}$. 
Asymptotics of $m$

$G$ a torus. $(M^{2d}, \omega, \mu, L)$ prequantized Hamiltonian $G$-space

$$\text{index}_G(D_{L^k}) = \sum_{\lambda \in \Lambda} m(k, \lambda) \chi_{\lambda}$$

$$\Theta(m; k) = \sum_{\lambda \in \Lambda} m(k, \lambda) \delta_{\lambda/k}$$

**Theorem**

$$\Theta(m; k) \sim k^d \sum_{n=0}^{\infty} \frac{1}{k^n} DH_M(Td_n(M))$$

Equivariant Todd class, expanded in homogeneous terms:

$$Td(M)(X) = \sum_{n=0}^{\infty} Td_n(M)(X)$$
Piecewise quasi-polynomial functions

Fix lattice $\Lambda \subset V$

$P$ rational polyhedron, $\sigma \in V_\mathbb{Q}$

$$C_{P,\sigma} = \{(t, tv + \sigma) | t > 0, v \in P\}$$

characteristic function: $[C_{P,\sigma}]$

**Definition**

$m : \mathbb{Z}_{>0} \times \Lambda \rightarrow \mathbb{C}$ is piecewise quasi-polynomial if

$$m = \sum_{P,\sigma} q_{P,\sigma} [C_{P,\sigma}]$$

where $q_{P,\sigma}(k, \lambda)$ quasi-polynomial and $\{P + [0,1]\sigma\}$ is locally finite.
The function $m = [C_P] + [C_{P,\sigma}]$ for $\Lambda = \mathbb{Z} \subset \mathbb{R} = V$, $P = [0, 1]$, and $\sigma = 4$. On the light gray region $m = 1$, and on the dark gray region $m = 2$. 
Family of measures

\[ \Theta(m; k) = \sum_{\lambda \in \Lambda} m(k, \lambda) \delta_{\lambda/k} \]

Support of \( \Theta(q[C_P]; k) \), \( k = 3, 6, 12 \), with \( \Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2 = V \), and \( P = \{ x \geq 0, y \geq 0, x + y \leq 1 \} \).
Support of $\Theta(q[C_P,\sigma]; k)$, $k = 3, 6, 12$, with $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2 = V$, $\sigma = \left(\frac{3}{2}, \frac{9}{2}\right)$, and $P = \{x \geq 0, y \geq 0, x + y \leq 1\}$. 
\( \Theta(k) \) a sequence of distributions \( k = 1, 2, 3, ... \)

**Definition**

\( \Theta(k) \) *admits an asymptotic expansion* if there is a \( j \in \mathbb{Z} \) and distributions \( u_n(k) \), *periodic in \( k \)*, such that for all \( N \) and all test functions \( \varphi \),

\[
\langle \Theta(k), \varphi \rangle = k^j \sum_{n=0}^{N} \frac{1}{k^n} \langle u_n(k), \varphi \rangle + o(k^{j-N})
\]

\( \Theta(k) \sim \mathcal{A}(k), \quad \mathcal{A}(k) = k^j \sum_{n=0}^{\infty} \frac{1}{k^n} u_n(k) \)
Theorem (LPV)

For any piecewise quasi-polynomial $m$, the sequence $\Theta(m; k)$ admits an asymptotic expansion $A(m; k)$.

If $m$ has compact support, asymptotic expansion becomes exact when both sides are paired with polynomials. (Ehrhart’s theorem on counting lattice points in $kP$ is a special case.)

- Series of reductions: general case $\leadsto$ single polyhedron $\leadsto$ single cone $\leadsto$ 1-dimensional cone.
- 1-dimensional case is the Euler-Maclaurin formula.
Example with $V = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $P = [0, 1]$

$$m(k, \lambda) = \begin{cases} 1 & \text{if } 0 \leq \lambda \leq k \\ 0 & \text{else.} \end{cases}$$

$$\Theta(m; k) = \sum_{\lambda=0}^{k} \delta_{\lambda/k}$$

$$A(m; k) = k \mu_{[0,1]} + \frac{1}{2} (\delta_0 + \delta_1) + k \sum_{n=2}^{\infty} \frac{B_n}{n!k^n} (-1)^{n-1}(\delta_1^{n-1} - \delta_0^{n-1})$$

$\mu_{[0,1]}$ Lebesgue on $[0, 1]$, $B_n$ Bernoulli numbers,

$\delta_x^{(r)}$ $r^{th}$ derivative of $\delta_x$
Example with $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

$$P = \{(\lambda_1, \lambda_2)|\lambda_i \geq 0, \lambda_1 + \lambda_2 \leq 1\}$$

$$m(k, \lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } (\lambda_1, \lambda_2) \in kP \\ 0 & \text{else.} \end{cases}$$

Let $\partial_0 P = P \cap \{\lambda_1 + \lambda_2 = 1\}$, $\partial_1 P = P \cap \{\lambda_2 = 0\}$, $\partial_2 P = P \cap \{\lambda_1 = 0\}$ closed 1-dim faces.

Leading and sub-leading terms of $A(m; k)$:

$$k^2 \mu_P, \quad \frac{1}{2} k (\mu_{\partial_1 P} + \mu_{\partial_2 P} + \mu_{\partial_0 P}).$$
Kernel of $m \mapsto A(m)$

How much information does $A(m)$ contain?

Not perfect. Example: $V = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $m(k, \lambda) = (-1)^{\lambda} \Rightarrow \Theta(m; k) \sim 0$.

With a slightly stronger local finiteness condition, we can prove that obvious generalizations of the above example are responsible for the whole kernel.

Corollary: $A$ is injective on subspace of $m$'s given by finite sums only involving compact $P$'s.
$g \in \text{Hom}(\Lambda, U(1))$ of finite order

$$(g \cdot m)(k, \lambda) = g^\lambda m(k, \lambda)$$

**Theorem (LPV)**

*If $A(g \cdot m) = 0$ for all $g$ then $m = 0$.***

**Application [PV]:** functoriality of quantization (symplectic or spin-c) under restriction to subgroups in the non-compact setting.

("Proof": restriction is easy for (twisted) DH distributions. Use above theorem (also Berline-Vergne and Kirillov formulas) to prove result.)
Comments on the proof

Step 1: piecewise quasi-polynomials have ‘germs’: if 
\[ m = \sum q_{P,\sigma}[C_{P,\sigma}] \], the ‘germ’ at \( v \in V \):

\[ T_v m = \sum q_{P,\sigma}[C_{T_v P,\sigma}] \]

\( T_v P \) the tangent cone to \( P \) at \( v \), and \( m = 0 \iff T_v m = 0 \forall v \in V \)

Step 2: Say \( T_0 P \) is a single pointed cone (for simplicity). Fourier transform \( \mathcal{F}(\Theta(T_0 m; k)) \) is the boundary value of a meromorphic function, e.g.

\[ \frac{1}{1 - e^{iz}} \quad \text{where} \quad z = x/k + i\epsilon, \quad \epsilon \to 0^+ \]

Step 3: \( \mathcal{F}(A(T_0 m; k)) \) obtained by Laurent expansion at 0. If this vanishes then original function vanishes.
Outlook

- Modest generalizations with auxiliary vector bundles
- Simplify aspects of the proof of \([Q, R] = 0\) in the singular case
- Re-examine spin-c \([Q, R] = 0\) (due to Paradan-Vergne)? Relation with symplectic \([Q, R] = 0\) unclear. Good setting encompassing both?

Thanks!