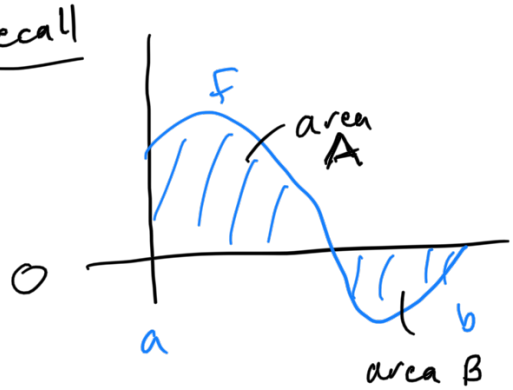
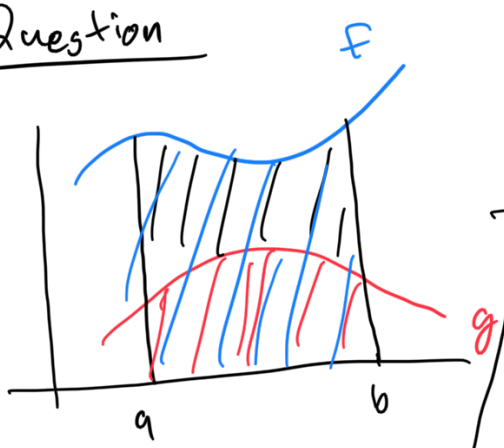
Lec 3/8: Area between curves

Recall



$$\int_a^b f(x)\,dx = \text{net signed area}$$
$$= A - B$$

Question

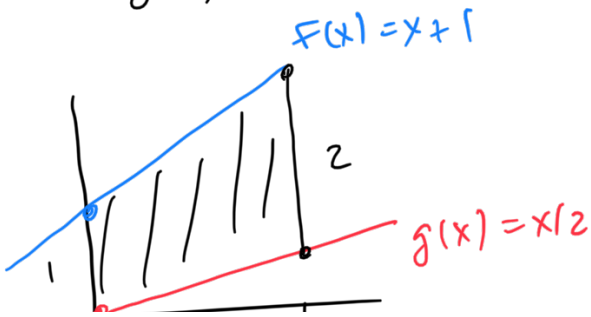


What is area between f and g over the interval $[a,b]$?

If $\underline{\underline{f \geq g}}$ on $[a,b]$ then

area between f and g over interval $[a,b]$ $= \int_a^b (f(x) - g(x))$

Ex Find area between $f(x) = x+1$ and $g(x) = x/2$ over the interval $[0,2]$.



$$\text{Area} = \int_0^2 \left(f(x) - g(x)\right) dx$$
$$= \int_0^2 \left(x+1 - x/2\right) dx$$

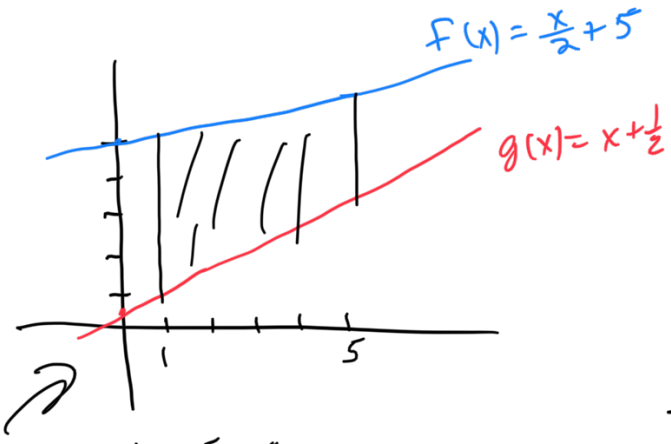
$$= \int_0^2 \left(\frac{x}{2}+1\right) dx$$

(Check: Area $\frac{3}{2}+2=3$)

$$= \frac{x^2}{4} + x \Big|_0^2 = \frac{2^2}{4}+2-(0+0$$

$$= 1+2 = 3$$

<u>Poll</u>: Find area between $f(x)=\frac{x}{2}+5$ and $g(x)=x+\frac{1}{2}$ over interval $[1,5]$.



$f(5)=\frac{5}{2}+5$

$g(5)=5+\frac{1}{2}$

$f(5)>g(5)$

$\Rightarrow f \geq g$ on $[1,5]$

Area

$$= \int_1^5 (f(x)-g(x))dx$$

$$= \int_1^5 \left(\left(\frac{x}{2}+5\right)-\left(x+\frac{1}{2}\right)\right)dx$$

$$= \int_1^5 \left(-\frac{x}{2}+\frac{9}{2}\right)dx$$

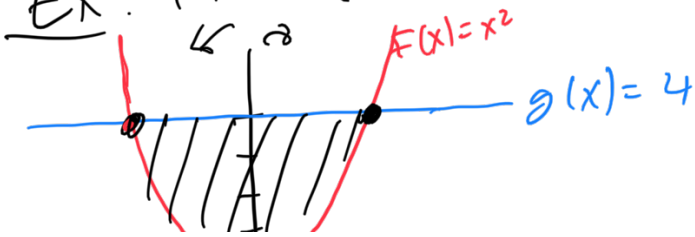
$$= -\frac{x^2}{4}+\frac{9}{2}x \Big|_1^5$$

$$= -\frac{5^2}{4}+\frac{9\cdot 5}{2}-\left(-\frac{1}{4}+\frac{9}{2}\right)$$

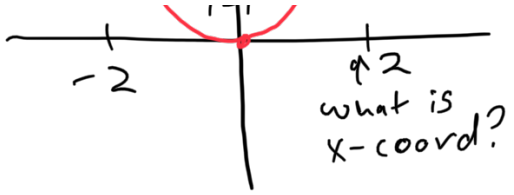
$$= -\frac{25}{4}+\frac{90}{4}+\frac{1}{4}-\frac{18}{4}$$

$$= \frac{65-17}{4} = \frac{48}{4} = 12$$

<u>Ex</u>: Find area enclosed by $f(x)=x^2$ and $g(x)=4$



Intersection points $\Leftrightarrow$ x such that $f(x)=g(x)$



$x^2 = 4$

$x = \sqrt{4} = \pm 2$

$g \geq f$ on $[-2,2]$

$$\text{Area enclosed} = \int_{-2}^{2} \big(g(x) - f(x)\big)\,dx$$

$$= \int_{-2}^{2} (4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2}$$

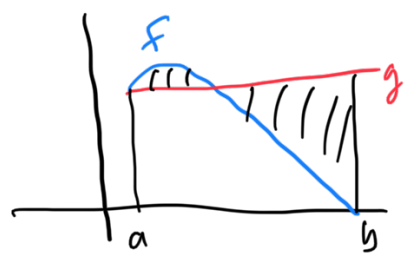
$$= \left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 16 - \frac{16}{3} = \frac{3 \cdot 16 - 16}{3} = \frac{2 \cdot 16}{3}$$

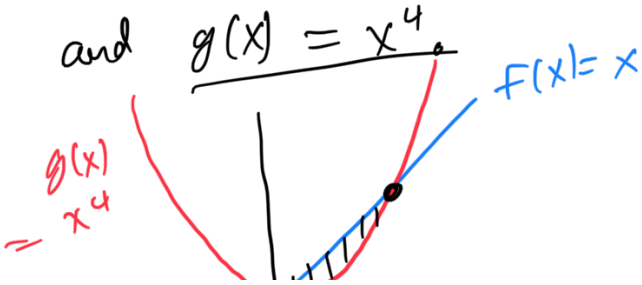
$$= \frac{32}{3}$$

WARNING:



Neither $\int_a^b (f(x) - g(x))\,dx$, nor $\int_a^b (g(x) - f(x))\,dx$ is equal to the area

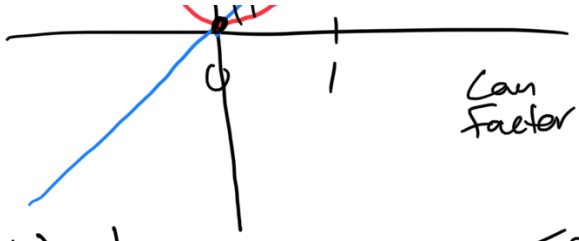
<u>Poll</u>: Find area enclosed by $f(x) = x$ and $g(x) = x^4$.



Intersection points:

$f(x) = g(x)$

$x = x^4$



Can factor $\rightarrow$ $0 = x^4 - x$

$0 = x(x^3 - 1)$

Real solutions:

$x = 0$

$x = 1$

$f(\frac{1}{2}) = \frac{1}{2}$

$g(\frac{1}{2}) = (\frac{1}{2})^4 = \frac{1}{16}$

so $f(1/2) > g(1/2)$

Note: $f \geq g$ on $[0,1]$

$$\text{Area enclosed} = \int_0^1 (f(x) - g(x))\,dx$$

$$= \int_0^1 (x - x^4)\,dx$$

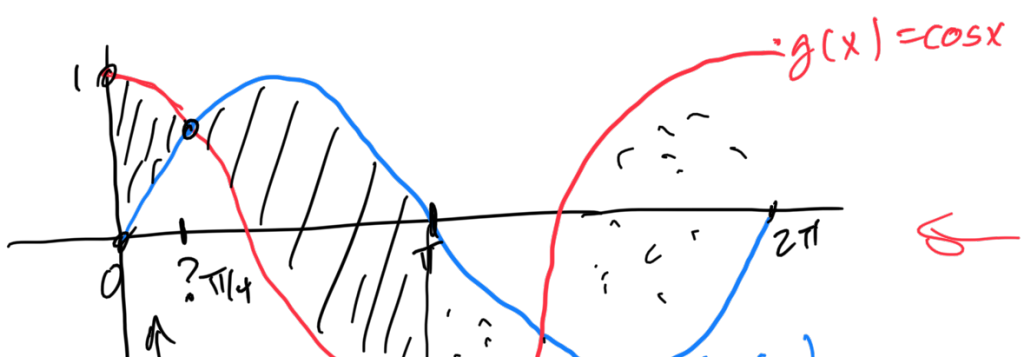
$$= \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{5}\right) - (0 - 0)$$

$$= \left(\frac{5}{10} - \frac{2}{10}\right) = \frac{3}{10}$$

Crossing Curves

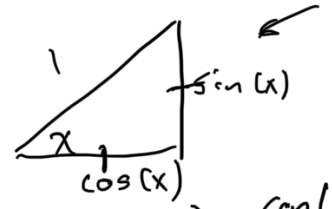
Ex: Find area enclosed between graphs of $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over interval $[0, \pi]$



First, find intersection pt

$$f(x) = g(x)$$

$$\sin(x) = \cos(x)$$



$\Rightarrow x = \pi/4$ (only solution in $[0, \pi]$)

$\sin(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

Area enclosed over $[0, \pi]$

= Area enclosed over $[0, \pi/4]$

+ Area " " $[\pi/4, \pi]$

$$= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\,dx$$

$$= (\sin(x) + \cos(x))\Big|_0^{\pi/4} + (-\cos(x) - \sin(x))\Big|_{\pi/4}^{\pi}$$

$$= (\sin \pi/4 + \cos \pi/4) - (\sin(0) + \cos(0))$$

$$+ (-\cos \pi - \sin \pi) - (-\cos \pi/4 - \sin \pi/4)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$+ (1 - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= (\sqrt{2} - 1) + 1 + \sqrt{2}$$

$$= 2\sqrt{2}$$

In general, area between graphs of $f(x)$, g over interval a, b

$$= \int_a^b |f(x) - g(x)| \, dx$$

absolute value

e.g. area in previous Ex.

$$= \int_0^{\pi} |\sin x - \cos x| \, dx$$