Lec 3/8: Area between curves

Recall

\[ \int_a^b f(x) \, dx = \text{net signed area} = A - B \]

Question

What is area between \( f \) and \( g \) over the interval \([a, b]\)?

If \( f \geq g \) on \([a, b]\) then

\[ \text{area between } f \text{ and } g = \int_a^b (f(x) - g(x)) \, dx \]

Ex

Find area between \( f(x) = x + 1 \) and \( g(x) = x/2 \) over the interval \([0, 2]\).

\[ \text{Area} = \int_0^2 (f(x) - g(x)) \, dx = \int_0^2 (x + 1 - x/2) \, dx \]
\[
\text{Area} = \int_1^5 (f(x) - g(x)) \, dx \\
= \int_1^5 \left( \frac{x}{2} + 5 - \left( x + \frac{1}{2} \right) \right) \, dx \\
= \int_1^5 \left( \frac{x}{2} + \frac{9}{2} \right) \, dx \\
= \left[ \frac{x^2}{4} + \frac{9}{2} x \right]_1^5 \\
= \left( \frac{25}{4} + \frac{45}{2} \right) - \left( \frac{1}{4} + \frac{9}{2} \right) \\
= \frac{65}{4} - \frac{17}{4} = \frac{48}{4} = 12
\]

Ex: Find area enclosed by \( f(x) = x^2 \) and \( g(x) = 4 \)

Intersection points \( x \) such that 
\( f(x) = g(x) \)
$g$? F on $[-2,2]$.

Area enclosed = \[ \int_{-2}^{2} (g(x) - F(x)) \, dx \]
\[ = \int_{-2}^{2} (4 - x^2) \, dx \]
\[ = \left[ 4x - \frac{x^3}{3} \right]_{-2}^{2} \]
\[ = \left( 4(2) - \frac{8}{3} \right) - \left( 4(-2) - \frac{(-2)^3}{3} \right) \]
\[ = \left( 8 - \frac{8}{3} \right) - \left( -8 + \frac{8}{3} \right) \]
\[ = 16 - \frac{16}{3} = 3 \cdot \frac{16 - 16}{3} = \frac{2 \cdot 16}{3} \]
\[ = \frac{32}{3} \]

**WARNING:**

\[ \int_{a}^{b} (F(x) - g(x)) \, dx \]
\[ = \int_{a}^{b} (g(x) - F(x)) \, dx \]
is equal to the area.

Poll: Find area enclosed by $F(x) = x$ and $g(x) = x^4$.

Intersection points:
\[ F(x) = g(x) \]
\[ x = x^4 \]
Solve:
\[ f(x) = x^4 \]
\[ g(x) = x^3 - 1 \]

Real solutions:
\[ x = 0 \]
\[ x = 1 \]

Note: \( f \geq g \) on \([0, 1]\)

\[ \text{Area} = \int_{0}^{1} (f(x) - g(x)) \, dx \]

\[ = \int_{0}^{1} (x - x^4) \, dx \]

\[ = \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \]

\[ = \left( \frac{1}{2} - \frac{1}{5} \right) - (0 - 0) \]

\[ = \left( \frac{5}{10} - \frac{2}{10} \right) = \frac{3}{10} \]

**Crossing Curves**

**Ex.** Find area enclosed between graphs of \( f(x) = \sin(x) \) and \( g(x) = \cos(x) \) over interval \([0, \pi]\)

\[ g(x) = \cos(x) \]
First, find intersection point \( P \)

\[ f(x) = g(x) \]
\[ \sin(x) = \cos(x) \]

\[ \Rightarrow x = \frac{\pi}{4} \quad \text{(only solution in } [0, \pi] \text{)} \]

Area enclosed over \([0, \pi]\)

\[ = \text{Area enclosed over } [0, \frac{\pi}{4}] \]
\[ + \text{Area } \left[ \frac{\pi}{4}, \pi \right] \]

\[ = \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x)) \, dx + \int_{\frac{\pi}{4}}^\pi (\sin(x) - \cos(x)) \, dx \]

\[ = (\sin(x) + \cos(x)) \bigg|_0^{\frac{\pi}{4}} + (\cos(x) - \sin(x)) \bigg|_{\frac{\pi}{4}}^\pi \]

\[ = (\sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4})) - (\sin(0) + \cos(0)) \]

\[ + (\cos(\pi) - \sin(\pi)) - (\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})) \]

\[ = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) \]

\[ + (1 - 0) - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) \]

\[ = (\sqrt{2} - 1) + 1 + \sqrt{2} \]

\[ = 2\sqrt{2} \]
In general, area between graphs of $f(x), g(x)$ over interval $a, b$

$$= \int_a^b |f(x) - g(x)|\ dx$$

& absolute value

Eg, area in previous Ex.

$$= \int_0^{\pi} |\sin x - \cos x|\ dx$$