Lec 5/3

Poll: $\int_1^3 \frac{1}{(x-2)^2}\,dx$ evaluate

discontinuous at $x=2$
so Improper integral

$$= \int_1^2 \frac{1}{(x-2)^2}\,dx + \int_2^3 \frac{1}{(x-2)^2}$$

Original converges
$\Leftrightarrow$ both these converge

$$\int_1^2 \frac{1}{(x-2)^2}\,dx = \lim_{t\to 2^-} \int_1^t \frac{1}{(x-2)^2}\,dt$$

$(x-2)^{-2}$

$$= \lim_{t\to 2^-} -(x-2)^{-1}\Big|_1^t = \lim_{t\to 2^-}\left(-\frac{1}{(t-2)} - \left(-\frac{1}{1-2}\right.\right.$$

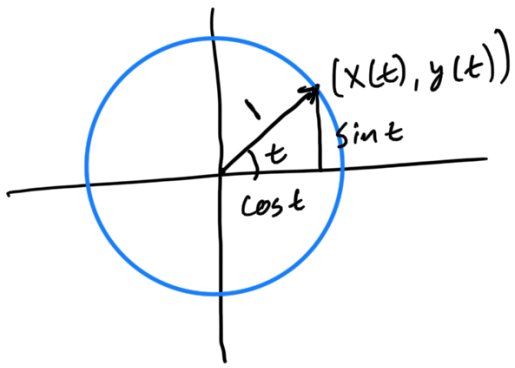
$$= -\frac{1}{(2-2)} + \frac{1}{1-2}$$ limit does not exis.

$\Rightarrow \int_1^2 \frac{1}{(x-2)^2}\,dx$ diverges

$\Rightarrow \int_1^3 \frac{1}{(x-2)^2}\,dx$ diverges

Parametric curves

Ex Parametrize unit circle by angle in radians

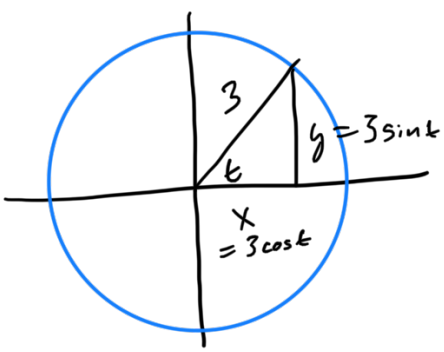


$x(t) = \cos(t)$

$y(t) = \sin(t)$

$0 \le t < 2\pi$

Poll: Give a parametrization of circle of radius 3 centered at origin.



$x(t) = 3\cos t$

$y(t) = 3\sin t$

$0 \le t < 2\pi$

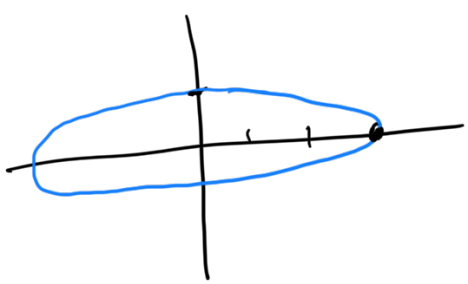
$\cos t = \frac{x}{3}$

$x = 3\cos t$

$\sin t = y/3$

$y = 3\sin t$

Ellipse



$x(t) = 3\cos t$

$y(t) = \sin t$

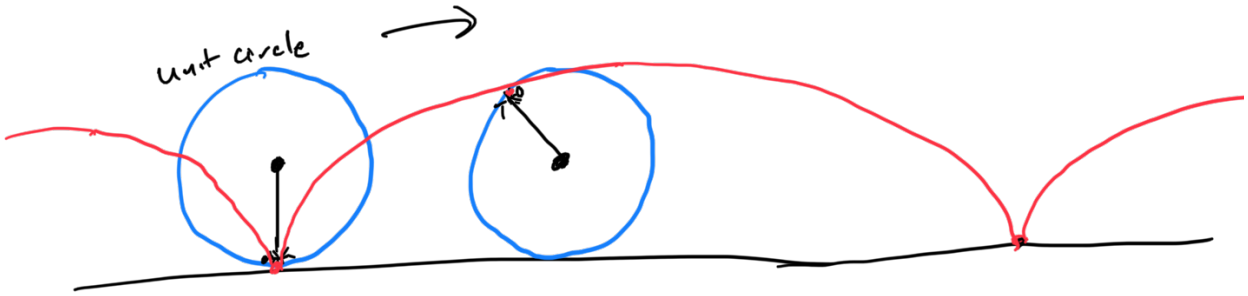
$0 \le t < 2\pi$

Cycloid

Q: what path does ant follow

A: Not a circular arc!

Called a cycloid

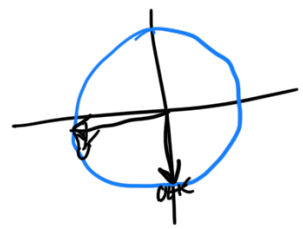


Center: $x(t) = t$

$y(t) = 1$

ant relative to center: $x(t) = -\sin(t)$

$y(t) = -\cos(t)$



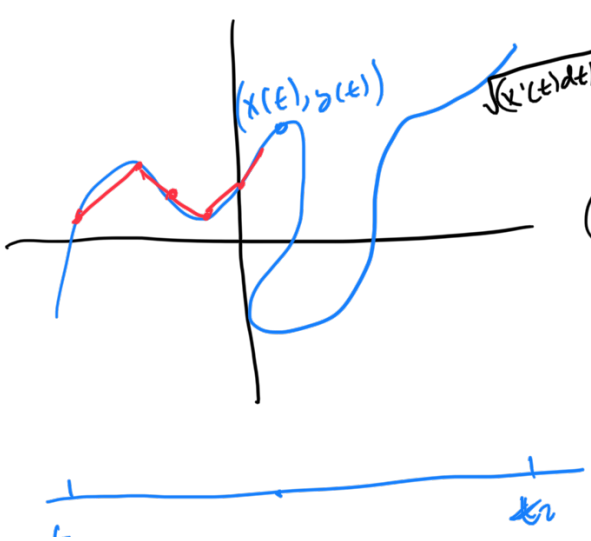
ant: $x(t) = t - \sin t$

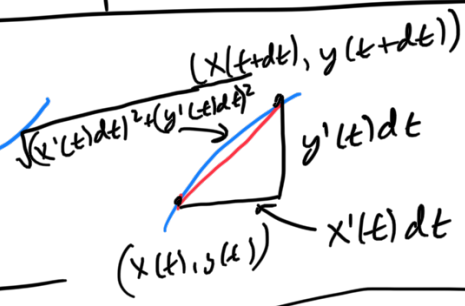
$y(t) = 1 - \cos t$

$-\infty < t < \infty$

← cycloid

Arc-length of a parametric curve





$x(t)$

$y(t)$

$t_1 \le t \le t_2$

Arc-length $= \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\, dt$

Ex unit Circle

$$x(t) = \cos t$$
$$y(t) = \sin t$$
$$0 \le t < 2\pi$$

Find arc-length.

$$x'(t) = \frac{d}{dt}\cos t = -\sin t$$
$$y'(t) = \frac{d}{dt}\sin t = \cos t$$

$$\text{Arc-length} = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

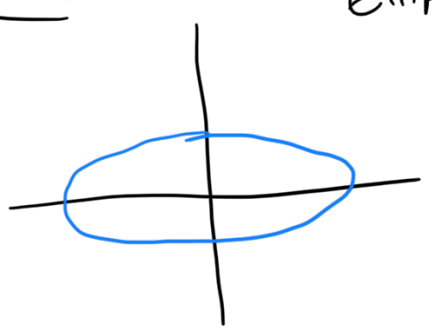
use trig identity (Pythag) $\rightarrow$ $= \int_0^{2\pi} \sqrt{1}\, dt$

$$= t\Big|_0^{2\pi} = 2\pi - 0 = 2\pi$$

Poll

Ellipse:

$$x(t) = 3\cos t$$
$$y(t) = \sin t$$
$$0 \le t \le 2\pi$$



Q: Write down an integral for arc-length [illegible] curve. Can you do this integral

of this curve.

$x'(t) = -3\sin t$

$y'(t) = \cos t$

$$\text{Arc-length} = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + \cos^2 t}\, dt$$

[Can do $\int \cos^i t \sin^j t$; above is <u>not</u> of this form]

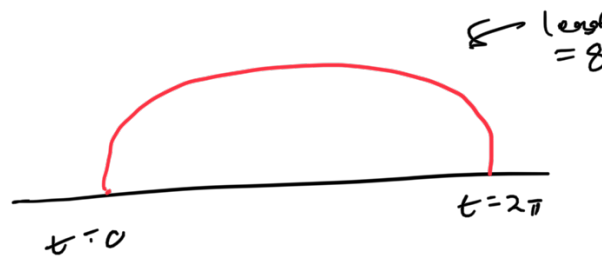
"Elliptic integral" $\rightsquigarrow$ impossible

Cycloid

$x(t) = t - \sin t$

$y(t) = 1 - \cos t$

$0 < t < 2\pi$



Compute arc-length

$x'(t) = 1 - \cos t$

$y'(t) = 0 + \sin t$

$$\text{Arc-length}: \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$$

$$\boxed{\begin{array}{l}\text{Trig halfangle} \\ \sin^2\frac{t}{2} = \frac{1-\cos(t)}{2}\end{array}}$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{2\left(2\cdot\sin^2\frac{t}{2}\right)}\, dt$$

$$= \int_0^{2\pi} 2\sin\frac{t}{2}\, dt$$

$$= -2\cdot 2\cos\frac{t}{2}\Big|_0^{2\pi}$$

$$= -4\cos\pi - (-4\cos(0))$$

$$= -4(-1) + 4 = 8$$