Homework 03 : MAT 364

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly. It is generally easy to convert a file from some other format, such as .docx, to .pdf.

1. Prove that $(-1,1) \subset \mathbb{R}$ and $(0,+\infty) \subset \mathbb{R}$ are homeomorphic.

2. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \cdot \sin(1/x) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

is continuous. You may use that the standard functions from calculus are continuous on their domains.

3. Give an example of two sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ (m,n can be whatever you want) for which X, Y are *not* homeomorphic, yet there exists a continuous, one-to-one, surjective function $f : X \to Y$. You do not need to rigorously prove that the X, Y that you choose are not homeomorphic (we don't have tools for this yet), but give an intuitive reason why they are not.

4. (*Kinsey 2.27*) Prove Lemma 2.26 (without using Theorem 2.24)

5. (*Kinsey 2.28*) Show that compactness is a topological property and give examples to show that closedness (as a subset of $\mathbb{R}^n$) and boundedness are not.

6. (*Kinsey 2.31*) Give examples of sets $A, B \subset \mathbb{R}^2$ which satisfy:

   (a) A and B are connected, but $A \cap B$ is not connected

   (b) A and B are connected, but $A - B$ is not connected

   (c) A and B are each non connected, but $A \cup B$ is connected.

(*Optional Challenge Problem for those who know about countable/uncountable sets*): Show that there is no way to write uncountably many numbers 8's in $\mathbb{R}^2$. That is, there does not exist an uncountable family of disjoint subsets X_α of $\mathbb{R}^2$ such that for each X_α there exists a homeomorphism $f_\alpha : E \to X_\alpha$, where E is a standard 8 in $\mathbb{R}^2$.