Homework 4: MAT 364

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly. It is generally easy to convert a file from some other format, such as .docx, to .pdf.

1. The one-dimensional case of the Brouwer Fixed Point Theorem states that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point, i.e. there exists some x in $[0, 1]$ with $f(x) = x$. Prove this directly from the Intermediate Value Theorem.

2. Find an example of a continuous function $f : (0, 1) \to (0, 1)$ that does not have a fixed point.

3. Let X be any set, and consider the discrete topology $\mathcal{T}$ on X, i.e. $\mathcal{T}$ consists of all subsets of X. Find a metric d on X such that $\mathcal{T}$ is the metric topology for this metric.

4. Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}^n$ of the form

$$\mathbb{R}^n - \bigcap_{f \in P} \{x : f(x) = 0\},$$

   where P is some collection of polynomials in n variables. Show that $\mathcal{T}$ defines a topology. (This is known as the *Zariski Topology*; it is of central importance in algebraic geometry.)

5. Let (X, d) be a metric space, and let $\mathcal{T}$ be the metric topology for $\mathcal{T}$. Prove that there exists a metric d' on X such that the metric topology for (X, d') is also $\mathcal{T}$, and (X, d') is *bounded* (i.e. for some $x \in X, r \in \mathbb{R}_+$, $X \subset \{y : d'(y, x) < r\}$).

6. (*Kinsey 3.1*) Prove (using the definitions of topology and basis from our lectures) that if $(X, \mathcal{T})$ is a topological space with a basis $\mathcal{B}$, then a set $U \subset X$ is open (i.e. $U \in \mathcal{T}$) if and only if U can be written as a union of elements of $\mathcal{B}$.