Homework 12: MAT 364

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly. It is generally easy to convert a file from some other format, such as .docx, to .pdf.

1. Let X, Y be topological spaces. Show that the notion of homotopy of continuous maps from $X \to Y$ is an equivalence relation, i.e. it is reflexive, symmetric, and transitive.

2. Show that the relation of homotopy equivalence for topological spaces is an equivalence relation.

3. Show that the Möbius strip is homotopy equivalent to the circle.

4. (Kinsey 9.2) Classify the letters of the alphabet up to homotopy equivalence. This does not need to be rigorous.

5. Show that if X, Y are homotopy equivalent topological spaces, then X is simply connected iff Y is simply connected.

6. Consider the projective plane, presented as

$$\mathbb{P}^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}/\sim,$$

where $x \sim y$ iff $x = y$, or $\|x\| = 1$ and $y = -x$ (i.e. take the closed disc and identity antipodal points on the boundary). Consider the continuous function $\gamma : S^1 \to \mathbb{P}^2$ given by

$$\gamma(\theta) = \begin{cases} (-1 + 4\theta, 0) & \text{if } \theta \in [0, 1/2] \\ (-1 + 4(\theta - 1/2), 0) & \text{if } \theta \in (1/2, 1). \end{cases}$$

Here we are thinking of S^1 as $[0, 1]$ with endpoints glued together. Show that γ is homotopic to a constant map $S^1 \to \mathbb{P}^2$.

Optional Challenge Problem: Can one embed uncountably many disjoint copies of the letter Y into $\mathbb{R}^2$?