Homework 13 : MAT 364

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly. It is generally easy to convert a file from some other format, such as .docx, to .pdf.

1. A subset S of $\mathbb{R}^n$ is said to be *convex* if for all $x, y \in S$, the line segment connecting x to y is contained in S. Show that any such convex S is simply connected.

2. Explain why the proof I gave in lecture for the $n = 2$ case of the Brouwer fixed point theorem does not easily generalize to $n = 3$ or higher (the result is still true in higher dimensions, but different techniques are needed to prove it).

3. Prove that composition of homotopy classes of based loops is associative, i.e. if $[\gamma_1], [\gamma_2], [\gamma_3]$ are homotopy classes of based loops on (X, p), then

$$([\gamma_1] * [\gamma_2]) * [\gamma_3] = [\gamma_1] * ([\gamma_2] * [\gamma_3]) .$$

4. Show that composition of loops (not up to homotopy) is *not* associative, i.e. find a space X with a base point p, and based loops $\gamma_1, \gamma_2, \gamma_3$ such that

$$(\gamma_1 * \gamma_2) * \gamma_3 \neq \gamma_1 * (\gamma_2 * \gamma_3) .$$

Optional challenge problem: If Jane runs a 10 mile race in 80 minutes, must there be some mile that Jane runs in exactly 8 minutes? Now suppose the race is 10.5 miles, and Jane runs it in 84 minutes; must there be some mile that Jane runs in exactly 8 minutes?