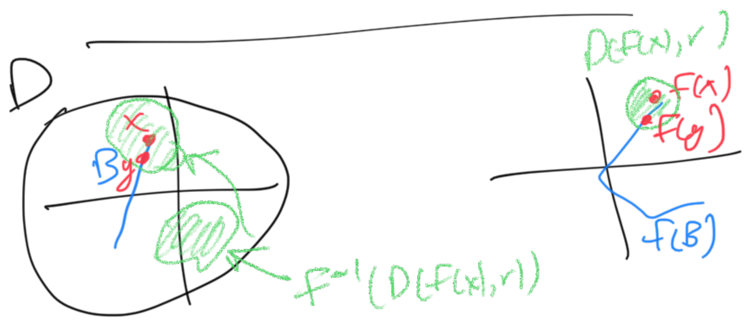
Continuity (continued)

Thm Let $D \subset \mathbb{R}^n$, $R \subset \mathbb{R}^m$, $f: D \to R$

f is continuous $\iff$ for any $x \in D$ that's limit point of $B \subset D$, $f(x)$ is limit pt of $f(B)$



Pf ($\Rightarrow$)

$x \in D$, limit point B

Want: Any open neighborhood $D(f(x), r)$ meets $f(B)$

Assuming $C \subset R$ is open, then $f^{-1}(C)$ open in D

know: $f^{-1}(D(f(x), r))$ is open in D (by continuity of f)

$x \in f^{-1}(D(f(x), r))$

Since x is limit point of B, know there's some $y \in B$, $y \in f^{-1}(D(f(x), r))$

means $f(y) \in D(f(x), r)$

$f(y) \in f(B)$

So $D(f(x), r)$ meets $f(B)$. ✓

Composition of Functions

$f: D \to R$, $g: R \to S$

$D \to S$

then $g \circ f$ function

$$g \circ f(x) = g(f(x))$$

<u>Ex</u>: $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$, $g(x) = x^3$

$$g \circ f(x) = g(f(x)) = g(x^2) = (x^2)^3 = x^6$$

<u>Identity</u> $1_D : D \to D$

$1_D(x) = x$ for all $x \in D$

<u>Def</u>: $f: D \to R$, say f is invertible if

$\exists g: R \to D$ s.t.

$g \circ f$ is 1_D, and $f \circ g$ is 1_R.

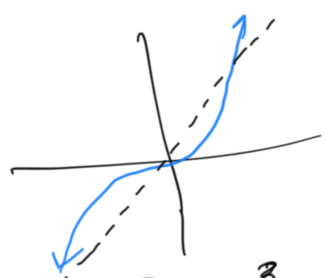
$f: \mathbb{R} \to \mathbb{R}$

<u>Ex</u>: $f(x) = x^3$ is invertible

inverse $g(x) = \sqrt[3]{x}$

$$g \circ f(x) = g(x^3) = \sqrt[3]{x^3} = x$$

$$f \circ g(x) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$



$f(x) = x^3$

$g(x) = \sqrt[3]{x}$

$f: \mathbb{R} \to \mathbb{R}$

<u>Non-ex</u>: $f(x) = x^2$ is <u>not</u> invertible

$f(1) = f(-1) = 1$

No good way to define $g(1)$

$$g(f(1)) = g(1) = g(f(-1))$$

$g \circ f$ cannot be identity

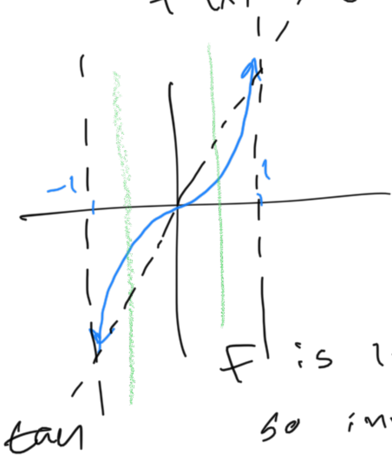
Rmk: f is invertible iff f is 1-1 (injective) <u>and</u> onto (surjective)

($f(x)=x^2$ is not 1-1, not onto)

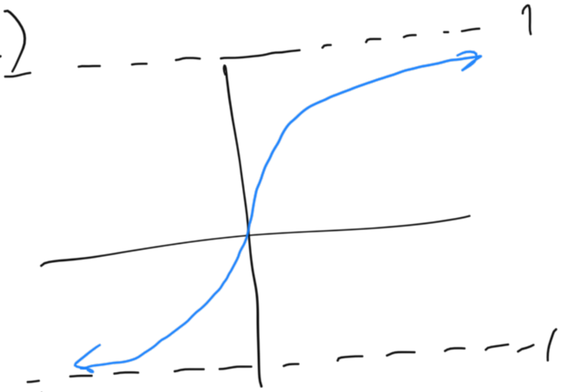
<u>Def</u> $f: D \to R$ say f is a <u>homeomorphism</u> (or topological equivalence) if f is continuous, invertible, and its inverse f^{-1} is continuous.

Ex: $f: (-1,1) \to \mathbb{R}$

$f(x) = \tan(\pi x/2)$



f is 1-1 and onto so invertible



arctan: $\mathbb{R} \to (-1,1)$