Compactness

Last time: $(-1,1)$ and $(-\infty,\infty) \subset \mathbb{R}$ are homeomorphic

$\uparrow$ bounded $\qquad$ $\uparrow$ unbounded

So boundedness is not a topological property

Def Set $A \subset \mathbb{R}^n$ is (sequentially) compact if every sequence $a_1, a_2, \dots \in A$ has a limit point in A.

Ex: Finite set $\underbrace{\{0,1,2\}}_{\text{Bounded: contained } D(0,3)} \subset \mathbb{R}$ $\quad$ Closed, so cpt

at least one of $0,1,2$ occurs infinitely often in $a_1, a_2, \dots$. That is a limit point.

Non-ex: $\overbrace{(0,1)}^{\text{Bounded } D(0,1)}$ is not compact. $\quad$ Not closed: limit pts $0,1$ $\quad$ Not cpt

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ converges to 0

only limit pt of the seq.

$0 \notin (0,1)$

Thm (Heine-Borel) $A \subset \mathbb{R}^n$

A is compact $\iff$ A is closed and bounded in $\mathbb{R}^n$

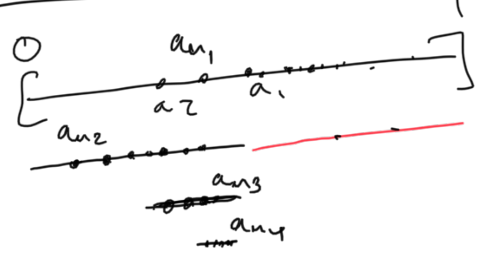
Tools for pf

Lemma 1: $I = [0,1] \subset \mathbb{R}$ is compact

Pf $a_1, a_2, \dots \in I$

Want: seq has a limit pt.

Infinitely many i s.t. a_i lies in left-half (for exampl



$n_1 < n_2 < n_3 < \dots$

$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$

Claim: $\{a_{n_i}\}$ is a Cauchy sequence

$$|a_{n_i} - a_{n_j}| \le \frac{1}{2^{n_i}}$$

$$j \ge i$$

"Completeness of real numbers" means every Cauchy seq. converges to unique limit

So $\lim_{i\to\infty} a_{n_i} = a$

Since $[0,1]$ is closed, so $a \in [0,1]$

in particular a is limit pt of $a_1, a_2, \ldots$ $\boxed{//}$

Lemma 2: If A, B are compact, ($A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$)

then $A \times B$ is compact

$$:= \{(a,b) : a \in A,\ b \in B\} \subset \mathbb{R}^{n+m}.$$

Pf: Start w/ $\{(a_k, b_k)\}_{k=1}^{\infty}$ seq in $A \times B$.

$a_1, a_2, \ldots \in A$

Compactness of $A \Rightarrow$ $\underline{a_{n_1}, a_{n_2}, \ldots}$ converges to $\underline{\text{some}}$ $a \in A$.

$b_{n_1}, b_{n_2}, \ldots \in B$

Compactness of $B \Rightarrow b_{n_{m_1}}, b_{n_{m_2}}, \ldots$ converges to some $b \in B$

Now $(\underline{a_{n_{m_i}}}, b_{n_{m_i}}) \to (a,b)$ as $i \to \infty$

$\in A \times B$

so $A \times B$ cpt $\boxed{//}$

Lemma 3: $I^n := [0,1]^n \subseteq \mathbb{R}^n$ is compact

Pf: $n=2$, $I^2 = I \times I$

I is cpt by Lem 1

$I^2 = I \times I$ is cpt by Lem 2.

$n=3 \quad I^3 = I^2 \times I$

$\vdots$

Pf of Heine-Borel $(\Leftarrow)$

Assume A closed + bounded

Want: A cpt.

Bounded: $A \subset B(0,R) \subset [-R,R]^n$.

$[-R,R]^n$ is cpt (for same reason I^n is: Lemma 3).

Exercise A is closed subset of a cpt set, then A itself is cpt

A is closed in $\mathbb{R}^n$, hence also in $[-R,R]^n$.

So by Exercise, A is compact.